Lecture 10: Quick review from previous lecture

- **Definition:** A **vector space** is a set V equipped with two operations:

  (1) (Addition) If $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} + \mathbf{w} \in V$.

  (2) (Scalar Multiplication) Multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

  For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

  (a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
  (b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
  (c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
  (d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.
  (e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
  (f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
  (g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Today we will discuss the subspace.

- Quiz 3 (covers sec. 1.8, 1.9, 2.1, 2.2) will take place in the beginning of the class on Wed. 2/19

Example 8. We let V be the upper right quadrant of $\mathbb{R}^2$, i.e.

$$V = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

We define addition of vectors by:

$$(x, y) \oplus (w, z) = (xw, yz)$$

and define scalar multiplication by:

$$c \odot (x, y) = (x^c, y^c)$$

* Here we use these notations $\oplus, \odot$ to distinguish it from ordinary ones.
Is V a vector space?

(1) $(x,y), (w,z) \in V$, $(x,y) \oplus (w,z) = (xw, yz) \in V$

(2) $c \odot (x,y) = (x^c, y^c) \in V$.

(a) $(x,y) \oplus (w,z) = (xw, yz) = (w,z) \oplus (x,y)$

(b) Similar to (a).

(c) What's zero element? $(1,1)$

$\boxed{(x,y)} \oplus (w,z) = (w,z)$

zero element

$\| $

(xw, yz)

Thus, $(x,y) = (1,1)$.

$(1,1) \oplus (w,z) = (w,z) = (w,z) \oplus (1,1)$.

(d)

| Exercise

(g)

Thus, V is a vector space. #

2.2 Subspaces

Definition: If $W \subset V$ (that is, W is a "subset" of V) and W is a vector space under the same addition and scalar multiplication defined on V, then W is called a **subspace** of V.

✓ "Subspaces" are <u>vector spaces</u> that are embedded in larger vector spaces.

If we want to check if $W \subset V$ is a subspace of V, it is enough to check the following 3 conditions:

1. W must contain zero element of V
2. If $\mathbf{v}$ and $\mathbf{w}$ in W, then $\mathbf{v} + \mathbf{w} \in W$.
3. If $\mathbf{v} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in W$.

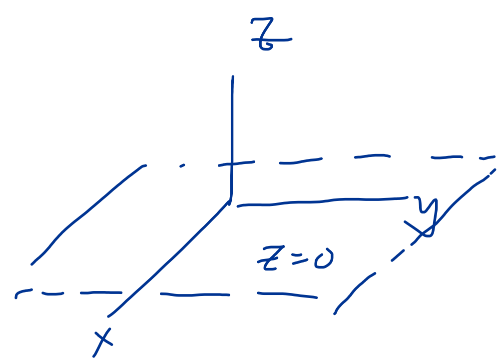
Example 1:

(1) $W = \{0\}$ is the trivial subspace of the vector space $\mathbb{R}^n$.

① $0 \in W$

② $0 + 0 = 0 \in W$

③ $c\,0 = 0 \in W$.



W $\shortparallel$ xy-plane

(2) $\{(x, y, 0)^T\}$ is a subspace of the vector space $\mathbb{R}^3$.

① $(0,0,0) \in W$

② $(x_1, y_1, 0), (x_2, y_2, 0) \in W$.

$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0) \in W$.

③ $c(x, y, 0) = (cx, cy, 0) \in W$

Example 2:

(1) $S = \{(x, y, 1)^T\}$ is NOT a subspace of the vector space $\mathbb{R}^3$.

$(0, 0, 0) \notin S$

$(x, y, 1) + (a, b, 1) = (\quad , \quad , 2) \notin S.$

(2) $S = \{x \geq 0,\ y \geq 0,\ z \geq 0\}$ is also NOT a subspaces of $\mathbb{R}^3$.

$(0,0,0) \in S.$

$-(1,2,3) = (-1,-2,-3) \notin S.$

(3) Another interesting example is the space of solutions to a linear homogeneous differential equation on $[a, b]$, for example,

$$S = \{u \overset{u \in \mathcal{F}([a,b]);}{}: \ u \text{ is the solution to } u''(x) + 2u'(x) + u(x) = 0\}.$$

Is S a subspace of $\mathcal{F}([a, b])$? (Yes) functions defined on $[a,b]$.

① $f=0$. $0'' + 2\,0' + 0 = 0$. $f=0 \in S$.

② $u, v \in S$, $u+v \overset{?}{\in} S$.

$$\left.\begin{array}{l} u'' + 2u' + u = 0 \\ v'' + 2v' + v = 0 \end{array}\right\} \xRightarrow{\text{add}} (u+v)'' + 2(u+v)' + (u+v) = 0$$

$\Rightarrow u+v \in S.$

③ $u \in S$, $c(\underbrace{u'' + 2u' + u}_{0}) = (cu)'' + 2(cu)' + (cu).$

Then $cu \in S.$

RK: 0 is essential for Example 2 (3) above.

2.3 Span and Linear Independence

Definition: Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors in a vector space V. If we take any scalars $c_1, \dots, c_n$, we can form a new vector in V as follows:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

An expression of this kind is known as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Example 1. If we have vectors $(1,2)^T$, $(-2,4)^T$ and $(5,-1)^T$ in $\mathbb{R}^2$, we can form the linear combination

$$\overset{c_1}{2}(1,2)^T \overset{c_2=-1}{-}(-2,4)^T + \overset{c_3=3}{3}(5,-1)^T = (19,-3)^T$$

$$(0,0)^T = 0\,(1,2)^T + 0\,(-2,4)^T + 0\,(5,-1)^T$$

Example 2. We observe that $0v = \overset{\text{zero element}}{\mathbf{0}}$ for each $v \in V$. Thus $\mathbf{0}$ is a linear combination of any nonempty subset of V.

Definition: If we fix some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V, we can consider the set of **all** of their linear combinations, This set is called the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, denoted

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

In other words,

$c_1 v_1 + \ldots + c_n v_n$

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \left\{ \sum_{i=1}^{n} c_i \mathbf{v}_i : c_1, \ldots, c_n \in \mathbb{R} \right\}$$

✓ In fact, $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.

① $\mathbf{0} = 0 v_1 \in \text{span}\{v_1, \ldots, v_n\}$

② $\sum_{i=1}^{n} c_i v_i, \quad \sum_{i=1}^{n} a_i v_i \in \text{span}\{v_1, \ldots, v_n\}$.

$\sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} \underline{(c_i + a_i)} v_i \in \text{span}\{v_1, \ldots, v_n\}$

③ $b \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} (b c_i) v_i \in \text{span}\{v_1, \ldots, v_n\}$.

Example 3. (1) Let $\mathbf{v}_1 = (1, 2, 3)$. What does $\text{span}\{\mathbf{v}_1\}$ consist of in $\mathbb{R}^3$?

a line, $\{ t(1,2,3) \mid t \in \mathbb{R} \}$

(2) What does $\text{span}\{(1, 0, 0), (0, 1, 0)\}$ consist of in $\mathbb{R}^3$?

xy-plane, $\{ t(1,0,0) + s(0,1,0) \mid t, s \in \mathbb{R} \}$.