Lecture 11: Quick review from previous lecture

- To check if $W \subset V$ is a subspace of V, it is enough to check the following 3 conditions:
  1. W must contain zero element of V,
  2. If $\mathbf{v}$ and $\mathbf{w}$ in W, then $\mathbf{v} + \mathbf{w} \in W$,
  3. If $\mathbf{v} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in W$.
- A linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

- We define the set of collecting all possible linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \left\{ \sum_{i=1}^{n} c_i\mathbf{v}_i : c_1, \ldots, c_n \in \mathbb{R} \right\}$$

Today we will discuss the linear independent (dependent).

- Quiz 3 (covers sec. 1.8, 1.9, 2.1, 2.2) will take place in the beginning of the class on Wed. 2/19

Remark:

- If $\mathbf{v}_1 \neq 0$ in $\mathbb{R}^3$, then span$\{\mathbf{v}_1\}$ is the line $\{c\mathbf{v}_1 : \ c \in \mathbb{R}\}$.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-zero vectors in $\mathbb{R}^3$ that are not parallel to each other (i.e. $\mathbf{v}_1 \neq c\mathbf{v}_2$ for any scalar c), then span$\{\mathbf{v}_1, \mathbf{v}_2\}$ defines a *plane*.

Example 4. If $\mathbf{v}_1 = \overset{c\neq 0}{c}\mathbf{v}_2$, then what is span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

Observe that

$$a v_1 + b v_2 = a(c v_2) + b v_2$$
$$= \underbrace{(ac + b) v_2}_{} \rightarrow \text{in span}\{v_2\}.$$

$$\text{span}\{v_1, v_2\} = \text{span}\{v_2\}$$
$$= \text{span}\{v_1\}.$$

This will lead us naturally into the topic of **linear independence**, which we'll see soon.

Example 5. Determine the span of $f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2$.

$$\mathrm{span}\{1, x, x^2\} = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$$

$$= P^{(2)}, \text{ polynomials of degree} \le 2.$$

In general, $\mathrm{span}\{1, x, \dots, x^n\} = P^{(n)}$. #

§ Linear Independence and Dependence

Definition: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space V, we say they are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all of which are zero, so that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are not linearly dependent, we say they are **linearly independent**.

In other words, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent** if the *only* linear combination $\sum_{i=1}^n c_i\mathbf{v}_i$ that is equal to $\mathbf{0}$ is when all the c_i's are equal to 0.

$c_1 v_1 + \cdots + c_n v_n = 0$

Example 5. (1) From Example 1, we have seen that

$$\underbrace{2}_{c_1=2}(1,2)^T \underbrace{-}_{c_2=-1}(-2,4)^T + \underbrace{3}_{c_3=3}(5,-1)^T \underbrace{-}_{c_4=-1}(19,-3)^T = 0^T$$

thus $(1,2)^T, (-2,4)^T, (5,-1)^T, (19,-3)^T$ are linearly dependent.

(2) Let's consider

$$a(1,2)^T + b(-2,4)^T = 0^T.$$

$a v_1 + b v_2 = 0$

$[v_1 \ \ v_2]\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

What are a, b?

$$a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ homogeneous linear system $A\vec{x} = \mathbf{0}$

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{(2)-2(1)} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 8 & 0 \end{array}\right)$$

$b = 0, \quad a = 0.$

or

$A = \begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix}$

$\det A = 8 \neq 0$

A is invertible (nonsingular)

Thus $Ax = 0$ has solution $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Thus $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right\}$ are linearly independent.

Example 6. We take the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -2 \\ -6 \end{pmatrix}.$$

Determine if they are linearly independent or not?

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \text{ Find } c_1, c_2, c_3.$$

$$\begin{pmatrix} v_1 & v_2 & v_3 \\ & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ homogeneous linear system.}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 2 & 2 & -2 & 0 \\ 3 & 4 & -6 & 0 \end{array}\right).$$

$$\xrightarrow[③-3①]{②-2①} \left(\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & 4 & -12 & 0 \\ 0 & 7 & -21 & 0 \end{array}\right)$$

$$\xrightarrow{③-\frac{7}{4}②} \left(\begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & 4 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{row echelon form}$$

$$\begin{pmatrix} 1 & -1 & 5 \\ 0 & 4 & -12 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Downarrow$

singular, $Ax = \vec{0}$ has nontrivial solution.

Thus, $\{v_1, \dots, v_3\}$ linearly dependent. #

$$4c_2 - 12c_3 = 0 \implies \boxed{c_2 = 3c_3}$$

$$c_1 - c_2 + 5c_3 = 0 \implies \boxed{c_1 = -2c_3.}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2c_3 \\ 3c_3 \\ c_3 \end{pmatrix}.$$

Taking $c_3 = 1$, $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$. $-2v_1 + 3v_2 + v_3 = \vec{0}$

Thus, $\{v_1, v_2, v_3\}$ are linearly dependent. #

[Example 6 continue...]

We can then conclude that

Theorem: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ and let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$:

(1) $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if there is a nonzero solution $\mathbf{x}$ to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(2) $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if the only solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is the trivial one, $\mathbf{x} = \mathbf{0}$.

(3) A vector $\mathbf{b} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ if and only if $A\mathbf{x} = \underbrace{\mathbf{b}}_{\parallel}$ is compatible (i.e., has at least one solution).

$$A\vec{x} = [v_1 \cdots v_k] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = c_1 v_1 + \cdots + c_k v_k.$$

Example 7. Suppose we take any four vectors in $\mathbb{R}^3$; call them $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$. Can they be linearly independent? For instance, we take the 4 vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \left(c_1 v_1 + \cdots + c_4 v_4 = \vec{0} \right)$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & 0 \\ 2 & 0 & -4 & 2 & 0 \\ -1 & 4 & 6 & 3 & 0 \end{array} \right).$$

$$\xrightarrow[\textcircled{3}+\textcircled{1}]{\textcircled{2}-2\textcircled{1}} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & 0 \\ 0 & -6 & -6 & -6 & 0 \\ 0 & 7 & 7 & 7 & 0 \end{array} \right)$$

$$\xrightarrow{\textcircled{3}+\frac{7}{6}\textcircled{2}} \left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & 0 \\ 0 & -6 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{row echelon form.}$$

$$c_2 + c_3 + c_4 = 0, \quad c_2 = -c_3 - c_4$$

$$c_1 + 3c_2 + c_3 + 4c_4 = 0$$

$$c_1 = 2c_3 - c_4.$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2c_3 - c_4 \\ -c_3 - c_4 \\ c_3 \\ c_4 \end{pmatrix}, \quad (2c_3 - c_4) v_1 + (-c_3 - c_4) v_2 + c_3 v_3 + c_4 v_4 = \vec{0}$$

$\{v_1, \ldots, v_4\}$ linearly dependent. #

- We can use the same logic to show the general fact: if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are k vectors in $\mathbb{R}^n$, then they must be linearly dependent if $k > n$.
- In other words, we have the following fact:

Fact: If $k > n$, then any set of k vectors in $\mathbb{R}^n$ is linearly dependent.