Lecture 12: Quick review from previous lecture

- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space V, we say they are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all of which are zero, so that

  $$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

  If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are not linearly dependent, we say they are **linearly independent**.

Today we will discuss Basis and Dimension.

- Quiz 3 (covers sec. 1.8, 1.9, 2.1, 2.2) will take place in the beginning of the class on Wed. 2/19

(k≤n)

Fact: A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is linearly independent if and only if the rank of $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$ is equal to k.

row echelon form

$$\begin{array}{l} k\{ \\ n-k\{ \end{array}\begin{bmatrix} * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

State the result another way:

"The vectors $\{v_1, \ldots, v_k\}$ are linearly independent iff the homogeneous linear system $A\mathbf{c} = 0$ has NO free variables."

$\{v_1, \ldots, v_{n-1}, v_n\}$. l dependent.

Fact: If $\mathbf{v}_n$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, then $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$.

*See also Example 4: If $\mathbf{v}_1 = c\mathbf{v}_2$, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\}$.

Write v_n as

$$v_n = c_1 v_1 + \cdots + c_{n-1} v_{n-1}.$$

Taking any vector v in $\text{span}\{v_1, \ldots, v_{n-1}, v_n\}$.

To see if v is in $\text{span}\{v_1, \ldots, v_{n-1}\}$.

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n v_n. \\ &= a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n (c_1 v_1 + \cdots + c_{n-1} v_{n-1}) \\ &= (a_1 + a_n c_1)\underline{v_1} + \cdots + (a_{n-1} + a_n c_{n-1})\underline{v_{n-1}}. \end{aligned}$$

Thus v is in $\text{span}\{v_1, \ldots, v_{n-1}\}$.

2.4 Basis and Dimension

> **Definition:**
>
> (1) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we say that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ **span** V.
>
> (2) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V and are linearly independent, we say that they form a **basis** of a vector space V.

*So a basis for a vector space V is a ① linearly independent set of vectors that ② span V.

Example 1. The "standard basis" of $\mathbb{R}^n$ consists of the n vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus, ① $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span $\mathbb{R}^n$, since any vector $\mathbf{x} = \underline{(x_1, \ldots, x_n)}^T$ can be written as:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

To check that ② $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent:

$$c_1 e_1 + \cdots + c_n e_n = \vec{0}$$

$$\underbrace{[e_1 \ e_2 \ \cdots \ e_n]_{n\times n}}_{= I_n} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = I^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, $\{e_1, \ldots, e_n\}$ are l. independent.

So, $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ #

A natural question is: can there be a basis of $\mathbb{R}^n$ with a different number of vectors (not n)?

The answer is no!

In fact, "any basis of $\mathbb{R}^n$ must have exactly n vectors."

$\{v_1, \ldots, v_n\}$

Fact 1: If V is any vector space that has a basis with n vectors, then any other basis must also have n vectors.

$\{w_1, \ldots, w_k\}$.

To show this, we'll first show that

$V = \mathrm{span}\{v_1, \ldots, v_n\}$.

Fact 2: If V has a spanning set of size n, call it $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then any set of k elements $\mathbf{w}_1, \ldots, \mathbf{w}_k$ with $k > n$ is linearly dependent. [We will find $c_1, \ldots, c_k$, not all zero so that $c_1 w_1 + \cdots + c_k w_k = 0$]

w_j is in $V = \mathrm{span}\{v_1, \ldots, v_n\}$.

$w_j = \sum_{i=1}^{n} a_{ij} v_i$. To find c_j, not all zero, so that

$$
\begin{aligned}
0 &= c_1 w_1 + \cdots + c_k w_k \\
&= c_1 \left(\sum_{i=1}^{n} a_{i1} v_i \right) + \cdots + c_k \left(\sum_{i=1}^{n} a_{ik} v_i \right) \\
&= \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{k} a_{ij} c_j \right)}_{= A\vec{c}} v_i .
\end{aligned}
$$

$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{n1} & \cdots & a_{nk} \end{pmatrix}_{n \times k}$, $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$.

Since $k > n$, we can find $\vec{c} \neq \vec{0}$ to $A\vec{c} = \vec{0}$.

Thus, it implies $\{w_1, \ldots, w_k\}$ l. dep. $\not\exists$

[Proof of Fact 1:]

$V = \text{span}\{v_1, \ldots, v_n\}$. we must have $\boxed{k \le n}$, o/w $\{w_1, \ldots, w_k\}$ l. dependent.

Similar, $V = \text{span}\{w_1, \ldots, w_k\}$, $\boxed{k \ge n}$, o/w $\{v_1, \ldots, v_n\}$ l. dependent

So $k = n$.

We have shown that if a vector space V has a basis with n elements, then any other basis must have n elements too.

In this case, we say that n is the **dimension** of V, and denote its dimension by $\dim V$.

Example 1:

- We showed that $\mathbb{R}^n$ has a basis with n elements (the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$), $\mathbb{R}^n$ is n-dimensional, or $\dim \mathbb{R}^n = n$.

- Let $\mathbf{v}_1 \neq 0$ in $\mathbb{R}^3$. Then $\text{span}\{\mathbf{v}_1\} = \{c\mathbf{v}_1 : c \in \mathbb{R}\}$. What's dimension and basis? $\{v_1\}$ 1

- Let $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-zero vectors in $\mathbb{R}^3$ that are not parallel to each other. What's dimension and basis of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

  2 $\{v_1, v_2\}$.