Lecture 13: Quick review from previous lecture

- A basis for a vector space V is a linearly independent set of vectors that span V.
- If V is any vector space that has a basis with n vectors, then any other basis must also have n vectors.

Today we will discuss the Kernel.

Example 2: The vector space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$. What is its dimension?

We saw $P^{(n)} = \text{span}\{1, x, \ldots, x^{n+1}\}$.

To see if $\{1, x, \ldots, x^n\}$, we only need to show $\{1, x, x^2, \ldots, x^n\}$ is l. independent.

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0.$$

Then $c_0 = 0, \ldots, c_n = 0$, which implies $\{1, x, \ldots, x^n\}$ l. independent

So, $\{1, x, \ldots, x^n\}$ is a basis of $P^{(n)}$.

$$\dim P^{(n)} = n+1. \#$$

$(\Leftarrow)$

$(\Rightarrow)$

Fact 3: The elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{x} \in V$ can be written uniquely as a linear combination of the basis elements:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n.$$

$(\Rightarrow)$. We know $\{v_1, \ldots, v_n\}$ is a basis of V, and then we want to show $x \in V$.

$x = c_1 v_1 + \cdots + c_n v_n$ is unique.

$$x = a_1 v_1 + \cdots + a_n v_n.$$

$$0 = x - x = (c_1 - a_1) v_1 + \cdots + (c_n - a_n) v_n.$$

Since $\{v_1, \ldots, v_n\}$ is a basis, $\{v_1, \ldots, v_n\}$ is l. independent. This leads to

$$c_1 - a_1 = 0, \ldots, c_n - a_n = 0.$$

Then $c_k = a_k$, $k = 1, \ldots, n$. $\#$

Example 3: Determining if $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for $\mathbb{R}^3$ (is equivalent to determining the matrix $A = [v_1 \ v_2 \ v_3]$ is nonsingular.)

$A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$ if A is nonsingular, then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff \{ v_1, v_2, v_3 \}$ is l. independent.

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{①②}]{\text{switch}} \begin{pmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{③ - \frac{1}{3}①} \begin{pmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\xrightarrow{③ - \frac{1}{3}②} \begin{pmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

$\det A = -1(3)(-1)(-\frac{1}{3})$
$= -1 \neq 0$

Thus, A is nonsingular, so $\{ v_1, v_2, v_3 \}$ is a basis of $\mathbb{R}^3$.

Example 4: Check if $p_1(x) = x^2 + 3x$, $p_2(x) = x - 1$, $p_3(x) = x$ form a basis for $\mathcal{P}^{(2)}$. How can we do this?

In a basis $\{1, x, x^2\}$, we can associate P_k to

$$p_1(x) = x^2 + 3x \longrightarrow \begin{array}{c} 1 \\ x \\ x^2 \end{array} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$p_2(x) = x - 1 \longrightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$p_3(x) = x \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Just to check if $\begin{pmatrix} 0 & -1 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is nonsingular (See EX3).

Yes, $\{ p_1, p_2, p_3 \}$ is a basis for $p^{(2)}$. #

2.5 The Fundamental Matrix Subspaces

§ Kernel and Image

x in $\mathbb{R}^n \xrightarrow{A} y$ in $\mathbb{R}^m$.

We can associate to a matrix $A = A_{m \times n}$ a subspace of $\mathbb{R}^n$, called the *kernel* or *null space* of A.

Definition: The **kernel** of A is the set of all solutions $\mathbf{x}$ to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. We denote the kernel of A by $\ker A$:

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Fact 1: $\ker A$ is a subspace of $\mathbb{R}^n$:

1. 0 is in $\ker A$.

2. $c \in \mathbb{R}$, x_1, x_2 in $\ker A$. To see if $x_1 + x_2 \in \ker A$? $Ax_1 = 0$, $Ax_2 = 0$.

$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0.$

So, $x_1 + x_2 \in \ker A$.

if $cx_1 \in \ker A$?

$A(cx_1) = cAx_1 = c0 = 0.$

$cx_1 \in \ker A$. $\square$

Let's observe that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to the equation $A\mathbf{x} = \mathbf{b}$, then what can we say about their difference, $\mathbf{x}_1 - \mathbf{x}_2$?

$Ax_1 = b$, $Ax_2 = b$.

$A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0.$

Thus, $x_1 - x_2$ is $\ker A$.