Lecture 14: Quick review from previous lecture

- Let A be $m \times n$ matrix. $\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.
- If V is any vector space that has a basis with n vectors, then any other basis must also have n vectors.
- The elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{x} \in V$ can be written uniquely as a linear combination of the basis elements:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n.$$

- if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to the equation $A\mathbf{x} = \mathbf{b}$, then

  $\mathbf{x}_1 - \mathbf{x}_2$ in kerA

  $Ax_1 = b$, $Ax_2 = b$

  $A(x_1 - x_2) = b - b = 0.$

Today we will discuss the Kernel and Image.

- M1 (Next Friday, 2/28) covers C1, C2, except 1.7, 2.5, 2.6.
- Practice Exam will be available on Sat. on <u>Canvas</u>

2.5 The Fundamental Matrix Subspaces

§ Kernel and Image

Fact 2: Suppose the linear system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}^\star$. Then
$\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$
$\Leftrightarrow \mathbf{x} = \mathbf{x}^\star + z$, where $z \in \ker A$.

($\Rightarrow$) We know x is solution $Ax=b$, and then we want to show $x = x^* + z$, $z \in \ker A$.

So, $x - x^*$ is $\ker A$.

Let $z = x - x^* \in \ker A$.

So, $x = x^* + z$, $z \in \ker A$.

($\Leftarrow$) $x = x^* + z$, $z \in \ker A$, $Ax^* = b$.

To show if x is a solution of $Ax=b$,

$$Ax = A(x^* + z) = Ax^* + Az = b + 0 = b.$$

Thus, x is a solution of $Ax=b$.

Remark. In other words, *any* solution to $A\mathbf{x} = \mathbf{b}$ can be generated by starting with a *particular* solution $\mathbf{x}^*$, and adding to $\mathbf{x}^*$ vectors in the kernel of A.

Fact 3: Suppose A is a matrix and the linear system $A\mathbf{x} = \mathbf{b}^*$ has a unique solution $\mathbf{x}$ for some right hand side $\mathbf{b}^*$.

Then $\ker A = \{\mathbf{0}\}$ is the trivial subspace, which means $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$ compatible with A.

$A_{m\times n}$, $\ker A = \{0\} \iff \text{rank } A = n$.

§ To construct a basis for $\ker A$.

Example 1. Suppose we have a 3-by-5 matrix A, and after bringing it to row echelon form we get:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Find a basis for $\ker A$. $= \{ x \in \mathbb{R}^5 \mid Ax = 0 \}$.

Let's solve the system $Ax = 0$, $x = (x_1, \ldots, x_5)^T$.

$Ax = 0$, $\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$ x_3, x_5 free variables.

3rd equation: $x_4 = -2x_5$.

2nd equation: $x_2 = x_3 - 2x_4$

$= x_3 + 4x_5$.

1st equation: $x_1 = -x_2 = -x_3 - 4x_5$.

So, each element of $\ker A$ can be expressed as

[Example 1. Continue...]

$$\begin{pmatrix} -x_3-4x_5 \\ x_3+4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix}, \quad x_3, x_5 \in \mathbb{R}.$$

$x_3=1,\ x_5=0$: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x_3=0,\ x_5=1$: $\begin{pmatrix} -4 \\ 4 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$

$$\begin{pmatrix} -x_3-4x_5 \\ x_3+4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = x_3\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -4 \\ 4 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

$\boxed{A_{3\times 5} \quad \dim\ker A = 2 = 5 - \underbrace{3}_{\text{rank } A}.}$

NOTE: $W=\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ l. independent, spans $\ker(A)$. Thus W is a basis for $\ker A$, $\dim(\ker A)=2$ #

Remark. In general, here's how to build a basis for the kernel. There will be one basis vector for each free variable, which is constructed by setting that free variable to 1, and all the other free variables to 0.

In particular, this tells us that

$$\dim\ker A = n - r,$$

where $A = A_{m\times n}$ and r is the rank of A (since there are $n-r$ pivots).

Definition: The **image** of the matrix A is the set containing of all images of A, that is,

$$\operatorname{img} A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}.$$

Fact 4: $\operatorname{img} A$ is a subspace of $\mathbb{R}^m$ since it is the span of the columns of A.

$A_{m\times n}$. $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}_{m\times n}$. $\{Ax\} = \{x_1 v_1 + \cdots + x_n v_n\}$
$= \operatorname{span}\{v_1, \ldots, v_n\}$.

§ **To construct a basis for img** A**.**

Let's first do this for a matrix A that is already in row echelon form. Then we'll see how to generalize this to any matrix A.

So let's take, as an example,

Example 2. Find the basis of img A, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Look at submatrix formed the columns with pivots

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow 3\times 3 \text{ nonsingular matrix}$$

So, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \\ 0 \end{pmatrix} \right\}$ is a basis for $\operatorname{img}(A)$.

$\dim \operatorname{img} A = 3$.

This same reasoning will work in general.

Fact 5:

- If A is a matrix in row echelon form, then the columns of A containing the pivots form a basis for $\operatorname{img} A$.
- In particular, the dimension of the image of A will be the number of pivots, i.e. the rank r of A. That is,

$$\dim \operatorname{img} A = r.$$

*In particular, the rank of r does not depend on how we perform Gaussian elimination (that is, which permutations we perform). It only depends on A itself.

To summarize: to find a basis for $\operatorname{img} A$, bring A to row echelon form by Gaussian elimination. The columns of A (the original matrix A) where the pivots occur are a basis for $\operatorname{img} A$. (We will see an Ex later)

We have already seen that the number of pivots, i.e. the rank of A, is equal to the dimension of the image of A:

$$\dim \operatorname{img} A = r$$

So far, we have shown that if r is the rank of A, then

$$\dim \operatorname{img} A = r, \quad \dim \ker A = n - r$$

§ Coimage of A and cokernel of A.

Definition: The **coimage** of A is the image of its transpose, A^T. It is denoted coimg A:

$$\operatorname{coimg} A = \operatorname{img} A^T = \{A^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

The **cokernel** of A is the kernel of its transpose, A^T. It is denoted coker A:

$$\operatorname{coker} A = \ker A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T\mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

These are the four fundamental subspaces of A: img A, ker A, coimg A, and coker A.

§ coimg A

Let's study coimg A first. It is the span of the columns of A^T, or equivalently the span of the **rows** of A. For this reason, it's also called the **row space** of A.

In principle, we could build a basis for coimg A the same way we learned how to do for img A, by performing Gaussian elimination on A^T and taking the columns of A^T with pivots.

Example 3: Similar to what we did in Example 2 for A:

If $A^T = \begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

then $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}$ is a basis for $\operatorname{img} A^T = \operatorname{coimg} A$.