Lecture 15: Quick review from previous lecture

A : m×n matrix

- The **kernel** of A is

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

- The **image** of the matrix A is the set containing of all images of A, that is,

$$\operatorname{img} A = \{\overset{\in \mathbb{R}^m}{A\mathbf{x}} : \mathbf{x} \in \mathbb{R}^n\}.$$

- The **coimage** of A is the image of its transpose, A^T. It is denoted coimg A:

$$\operatorname{coimg} A = \operatorname{img} A^T = \{A^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

- The **cokernel** of A is the kernel of its transpose, A^T. It is denoted coker A:

$$\operatorname{coker} A = \ker A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T\mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

Today we will discuss the kernel and image, coker, and coimg as well as inner product.

- Midterm 1 covers C1, C2, except 1.7, 2.5, 2.6.
- Practice Exam is on Canvas.

✓ However, there is an alternate way of building a basis for coimg A. This method will let us see a profound connection between " img A" and "coimg A".

Observation.

- The key observation is that the row operations of Gaussian elimination do not change the row space of A.
- The two operations we perform for Gaussian elimination are (a) adding a multiple of one row to another row; and (b) permuting the order of rows.
- Permuting rows obviously does not change the row span. And it is easy to see that adding a multiple of one row to another row does not change the span of the rows.
- Consequently, the row echelon matrix U resulting from Gaussian elimination will have exactly the same row space as the matrix A we started with.

Conclusion 1: The row echelon matrix U has exactly the same row space as the original matrix A.

coimg A = coimg U.

If a matrix is in row echelon form, the nonzero rows are linearly independent, and consequently form a basis for the row space.

Conclusion 2: Therefore, to construct **a basis for the row space coimg** A, we can bring A to row echelon form using Gaussian elimination, and take the nonzero rows as the basis vectors.

Fact: If the rank of A is r, the basis we construct for coimg A will have r vectors. Thus,

$$\dim \operatorname{img} A = \dim \operatorname{coimg} A = r$$

§ coker A (A: $m \times n$ matrix)

A^T: $n \times m$ matrix

To build a basis for coker A, solve the n-by-m homogeneous system $A^T\mathbf{y} = \mathbf{0}$, and set each free variable to 1, and the others to zero.

In other words, apply the exact same procedure as for finding a basis for ker A.

What is the dimension of coker A? It is the number of free variables in $A^T\mathbf{y} = \mathbf{0}$. Since A^T has m columns and rank r, there are $m - r$ free variables, hence

Fact:

$$\dim \operatorname{coker} A = m - r$$

We can summarize what we've learned about the four fundamental subspaces in the following theorem, called the *Fundamental Theorem of Linear Algebra*:

Let A be an $m \times n$ matrix, and let r be its rank. Then

$$\dim \operatorname{coimg} A = \dim \operatorname{img} A = \operatorname{rank} A = \operatorname{rank} A^T = r,$$
$$\dim \ker A = n - r, \qquad \dim \operatorname{coker} A = m - r.$$

**Again, a very useful (and surprising) aspect of this theorem is that the column space and row space of A have the same dimension, equal to the rank r of A.

Example 4: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 4 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Find a basis for ker A, img A, coimg A, coker A, respectively.

$$A \xrightarrow[③-①]{②-2①} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & -1 & -2 \end{pmatrix} \xrightarrow{③-②} \begin{array}{cccc} \text{pivot} & \text{pivot} & & \\ \checkmark & \checkmark & & \\ \left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} & \begin{matrix} 2 \\ -1 \\ 0 \end{matrix} & \left.\begin{matrix} 3 \\ -2 \\ 0 \end{matrix}\right) \end{array}.$$

① Basis for img A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

② Basis for coimg A: $\left\{ (1, 1, 2, 3)^T, (0, -1, -1, -2)^T \right\}$.

③ Basis for ker A:

$Ax = 0$, $Ux = 0$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = -x_3 - x_4$$

$$\begin{aligned} x_1 &= -x_2 - 2x_3 - 3x_4 \\ &= -(-x_3 - x_4) - 2x_3 - 3x_4 \\ &= -x_3 - 2x_4 \end{aligned}$$

Thus, general solution for $Ax = 0$ is

$$\begin{pmatrix} -x_3 - 2x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \overset{x_3 = 1,\ x_4 = 0}{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}} + x_4 \overset{x_3 = 0,\ x_4 = 1}{\begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}}$$

So, $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\ker A$.

④ <u>Basis for coker A</u> $= \ker A^T$.

$$A^T = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}_{4\times 3} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

row echelon form.

$A^T x = 0$. $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

x_3 → free variable

$$x_2 = -x_3$$

$$x_1 = -2x_2 - x_3 = x_3.$$

General solution for $A^T x = 0$ is

$$\begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for coker A.

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3 Inner Products and Norms

3.1 Inner Products

§ Inner products in the Euclidean space $\mathbb{R}^n$

Definition: If $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ are any two vectors in $\mathbb{R}^n$, then we define their *inner product*, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$, by:
(dot product)

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

(row vector) (column vector)

Note that $\underbrace{[\quad]_{1\times n}}_{x^T} \underbrace{[\quad]_{n\times 1}}_{y} = [\ ]_{1\times 1}$

- $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x} \quad (\ = \mathbf{x}^T \mathbf{y})$$

  As in $\mathbb{R}^2$, if $\mathbf{x} = (x, y)^T$ is a vector, then the "Pythagorean Theorem" tells us that its length is given by $\sqrt{x^2 + y^2}$, and is denoted by

  $$\|\mathbf{x}\| = \sqrt{x^2 + y^2}.$$

- We will use this to define the length of vectors in $\mathbb{R}^n$ and denote the length of a vector $\mathbf{x}$ by

  $$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + \ldots + x_n^2}$$

  We call $\|\mathbf{x}\|$ the norm of $\mathbf{x}$.

- If $\mathbf{x} \neq 0$, then $\|\mathbf{x}\| > 0$. In addition, we also have

  $$\|\mathbf{x}\| = 0 \qquad \Leftrightarrow \mathbf{x} = 0.$$

- $\| cx \| = |c| \, \|x\|$.

Example.

1. If $\mathbf{x} = (1,1,1)^T$ and $\mathbf{y} = (-2,1,2)^T$, then find $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, $\langle \mathbf{x}, \mathbf{y}\rangle$ and also normalize $\mathbf{x}$ and $\mathbf{y}$. (Replace x by $\frac{x}{\|x\|}$) $\rightarrow \left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1$

$$\|x\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|y\| = \sqrt{(-2)^2+1^2+2^2} = \sqrt{9} = 3.$$

$$\langle x, y\rangle = \langle (1,1,1)^T, (-2,1,2)^T \rangle = 1$$

$$\frac{x}{\|x\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T.$$

$$\frac{y}{\|y\|} = \left(\frac{-2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T. \quad \#$$