Lecture 16: Quick review from previous lecture

- The **kernel** of A is
$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$
- The **image** of the matrix A is the set containing of all images of A, that is,
$$\operatorname{img} A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}. = \text{span}\{\text{columns of } A\}.$$
- The **coimage** of A is the image of its transpose, A^T. It is denoted coimg A:
$$\text{span}\{\text{columns of } A^T\} = \operatorname{coimg} A = \operatorname{img} A^T = \{A^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$
$$= \text{span}\{\text{rows of } A\}$$
- The **cokernel** of A is the kernel of its transpose, A^T. It is denoted coker A:
$$\operatorname{coker} A = \ker A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T\mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

Let A be an $m \times n$ matrix, and let r be its rank. Then
$$\dim \operatorname{coimg} A = \dim \operatorname{img} A = \operatorname{rank} A = \operatorname{rank} A^T = r,$$
$$\dim \ker A = n - r, \qquad \dim \operatorname{coker} A = m - r.$$

- In $\mathbb{R}^n$, the *inner product* is defined by:
  - $$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$
  - $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ $\qquad \|x\|^2 = \langle x, x \rangle.$
  - $\|cx\| = |c| \, \|x\|.$

Today we will discuss inner product and norms.

- Midterm 1 covers C1, C2, except 1.7, 2.5, 2.6.
- Practice Exam is on Canvas.

§ Abstract definition of general inner products

Definition: Let V be a vector space. An **inner product** on V is a functions that assigns every pairing two vectors $\mathbf{x}$ and $\mathbf{y}$ in V to obtain a real number, denoted

$$\langle \mathbf{x}, \mathbf{y} \rangle,$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in \mathbb{R}$, the following hold:

(1) Bilinearity:

$$\langle c\mathbf{u} + d\mathbf{v},\ \mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{w} \rangle + d\langle \mathbf{v},\ \mathbf{w} \rangle,$$
$$\langle \mathbf{u},\ c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{v} \rangle + d\langle \mathbf{u},\ \mathbf{w} \rangle,$$

(2) Symmetry: $\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{w},\ \mathbf{v} \rangle$,

3 ($\cancel{2}$) Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$. Moreover, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

A a vector space V equipped with a specific inner product is called an **inner product space**.

✓ We have already checked that the inner product on $\mathbb{R}^n$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

satisfies these three axioms.

Now let's take a look at some other inner product spaces.

Example. Let $C^0 = C^0(I)$ denote the vector space of continuous functions on an interval $I = [a, b]$, with the usual addition and scalar multiplication operations.

We can turn C^0 into an "inner product space" by defining the following *inner product*:

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx. \quad , \quad b > a.$$

*This is sometimes called the L^2 inner product (the "L" stands for "Lebesgue").

Let's check that this satisfies the defining properties of an inner product:

$f, g, h \in C^0$, $c, d \in \mathbb{R}$.

① Bilinearity:

$$\langle cf+dg, h\rangle = \int_a^b (cf+dg)(x)\, h(x)\, dx$$

$$= \int_a^b \left(cf(x) + dg(x)\right) h(x)\, dx.$$

$$= c\int_a^b f(x)\, h(x)\, dx + d\int_a^b g(x)\, h(x) dx$$

$$= c\langle f, h\rangle + d\langle g, h\rangle$$

$\langle f, cg+dh\rangle$ = Exercise $= c\langle f, g\rangle + d\langle f, h\rangle$.

② Symmetry:

$$\langle f, g\rangle = \int f(x)g(x)\,dx = \int g(x)f(x)\,dx$$

$$= \langle g, f\rangle.$$

③ **Positivity:**

$$\langle f, f \rangle = \int_a^b f^2(x)\, dx \geq 0.$$

Check: $\langle f, f \rangle = 0$ iff $f = 0$.

($\Leftarrow$) If $f = 0$, then $\int_a^b f^2\, dx = 0$

so $\langle f, f \rangle = 0$.

($\Rightarrow$) If $\langle f, f \rangle = 0$, we want to show $f = 0$.

$\int_a^b f^2(x)\, dx = 0$. Since f is continuous

we get f must be zero.

Thus, $\langle f, g \rangle = \int_a^b f g\, dx$ is an inner product. Then C^0 equipped with $\langle f, g \rangle$ is an inner product space.

§ The same vector space V can have many different inner products.

For example, while we originally equipped $\mathbb{R}^n$ with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$, we can also define "other" inner products on $\mathbb{R}^n$ as well. See discussions below.

Example. If $c_1, \ldots, c_n$ are positive numbers, we can define

$$\langle \mathbf{x}, \mathbf{y} \rangle = c_1 x_1 y_1 + \ldots + c_n x_n y_n = \sum_{i=1}^{n} c_i x_i y_i.$$

This is a legitimate inner product (check this as an exercise). It is called a weighted inner product, with weights $c_1, \ldots, c_n$.

① Bilinearity:

② Symmetry:

Exercises.

③ positivity:

$\langle x, x \rangle = c_1 x_1^2 + \cdots + c_n x_n^2 > 0$ if $x \neq \vec{0}$.

$\langle x, x \rangle = 0 \iff c_1 x_1^2 = 0, \ldots, c_n x_n^2 = 0$

$c_1 > 0, \ldots, c_n > 0$

$\iff x_1 = 0, \ldots, x_n = 0.$

Observe that while we can write the ordinary inner product on $\mathbb{R}^n$ as $\mathbf{x}^T \mathbf{y}$, we can write the "weighted inner product" as

$$\langle x, y \rangle = x^T \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} y = x^T D y, \quad D = \mathrm{diag}(c_1, \ldots, c_n), \text{ with } c_1 > 0, \ldots, c_n > 0.$$

– We can define an even more general class of inner products on $\mathbb{R}^n$, as follows:

Example. Take any n-by-n, nonsingular matrix A.

Now we define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A^T A \mathbf{y}.$$

Let's check that this is an inner product.

① Bilinearity : Exercise.

② Symmetry :

$$\begin{aligned} \langle x, y \rangle &= x^T A^T A y \quad \text{since } x^T A^T A y \text{ is a scalar.} \\ &= (x^T A^T A y)^T \\ &= y^T A^T A x \\ &= \langle y, x \rangle. \end{aligned}$$

③ Positivity :

$$\begin{aligned} \langle x, x \rangle &= x^T A^T A x \quad \text{Let } z = Ax \\ &= z^T z \\ &= \langle z, z \rangle \\ &= z_1^2 + \cdots + z_n^2 \end{aligned}$$

if $\langle x, x \rangle = 0 \iff z_1 = 0, \ldots, z_n = 0.$

$\iff z = 0$

$\iff Ax = 0$

$\iff x = 0$ since A is nonsingular

Thus it's an inner product.

Example. As an example of this kind of inner product in $\mathbb{R}^2$, let's define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = (x_1\ x_2) \overbrace{\begin{pmatrix} \sqrt{2} & -\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix}}^{A^T} \overbrace{\begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}}^{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1}$$

$$= 2x_1y_1 + 3(x_2 - x_1)(y_2 - y_1) \tag{2}$$

Remark. [Comparison:]

1. The norm of $(1, 3)^T$:
   - in usual inner product: $\sqrt{1+3^2} = \sqrt{10}$.
   - in inner product defined in (1):

     $\|(1,3)\|^2 = \langle (1,3), (1,3) \rangle = 14$

     $\|(1,3)\| = \sqrt{14}$.
2. The inner product of $(1, 1)^T$ and $(-1, 2)^T$:
   - in usual inner product: $\langle (1,1)^T, (-1,2)^T \rangle = -1+2 = 1$.
   - in inner product defined in (1):

$\langle (1,1)^T, (-1,2)^T \rangle = 2(1)(-1) + 3\overbrace{(1-1)}^{=0}(2+1)$

$= -2$. #

Summary.

- We saw many different inner products on $\mathbb{R}^n$, namely those of the form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T B \mathbf{y}$$

for a matrix $B = A^T A$ where A is nonsingular.

- When $B = I$, this includes the usual inner product $\sum_{i=1}^n x_i y_i$. Note that this usual inner product on $\mathbb{R}^n$ is called the *dot product*.

– Let's look at another example with weighted inner product.

Example. We can also define weighted inner products on $C^0(I)$. If $w(x)$ is any positive, continuous function, we can define

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx$$

It is a straightforward exercise to check that this is an inner product.

3.2 Inequalities

§ The Cauchy-Schwarz Inequality

In $\mathbb{R}^n$, we have seen from Calculus that the dot product between two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ can be geometrically characterized by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta,$$

where θ is the angle between the two vectors. Thus,

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

In fact,

Let V be an inner product space. Then the following **Cauchy-Schwarz inequality** holds:

$$|\langle\mathbf{v}\,,\,\mathbf{w}\rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\| \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

Thus, Cauchy-Schwarz inequality allows us to define the cosine of the angle θ between the two vectors $\mathbf{v}, \mathbf{w}$ in an inner vector space V:

$$\cos\theta = \frac{\langle\mathbf{v},\mathbf{w}\rangle}{\|\mathbf{v}\| \|\mathbf{w}\|}.$$

To see this, we know this ratio lies between -1 and 1, and defining the angle θ in this way makes sense.

Before showing Cauchy-Schwarz inequality , let's look at some examples.