Lecture 17: Quick review from previous lecture

- Let V be an inner product space. Then the following **Cauchy-Schwarz inequality** holds:

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\| \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

- A a vector space V equipped with a specific inner product is called an **inner product space**.
- $\langle \mathbf{v},\ \mathbf{w} \rangle$ is an **inner product** on V if the following hold:

  (1) Bilinearity:

$$\langle c\mathbf{u} + d\mathbf{v},\ \mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{w} \rangle + d\langle \mathbf{v},\ \mathbf{w} \rangle,$$
$$\langle \mathbf{u},\ c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{v} \rangle + d\langle \mathbf{u},\ \mathbf{w} \rangle,$$

  (2) Symmetry: $\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{w},\ \mathbf{v} \rangle$,

  (2) Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$. Moreover, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

<u>EX</u>: on $\mathbb{R}^n$, $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$ is an inner product.

Today we will discuss some inequalities and Norms.

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continuous functions on $[0,\pi]$.

Before showing Cauchy-Schwarz inequality, let's look at some examples.

Example 1. Consider $C^0[0,\pi]$, compute the inner product between $f(x) = \cos(x)$ and $g(x) = \sin(x)$. Also compute their angle θ.

$$\langle f,g\rangle = \int_0^{\pi} f(x)g(x)\,dx = \int_0^{\pi} \cos x \sin x\,dx$$

$$= \frac{1}{2}\sin^2 x \Big|_0^{\pi} = 0.$$

$$\cos\theta = \frac{\langle f,g\rangle}{\|f\|\|g\|} = 0. \quad \theta = \pi/2. \quad \#$$

On the other hand, consider $C^0[0,\pi/2]$, compute the inner product between $f(x) = \cos(x)$ and $g(x) = \sin(x)$. Also compute their angle θ.

$$\langle f,g\rangle = \int_0^{\pi/2} \cos x \sin x\,dx = \frac{1}{2}\sin^2 x\Big|_0^{\pi/2} = \frac{1}{2}(1-0) = 1/2.$$

$$\|f\|^2 = \langle f,f\rangle = \int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1+\cos(2x)}{2}\,dx$$

$$= \frac{1}{2}\,0\Big|_0^{\pi/2} = \pi/4.$$

$$\|g\|^2 = \langle g,g\rangle = \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2}\,dx.$$

$$= \pi/4$$

$$\cos\theta = \frac{\langle f,g\rangle}{\|f\|\|g\|} = \frac{1/2}{\pi/4} = \underline{\frac{2}{\pi}}$$

$$\theta = \cos^{-1}(2/\pi). \quad \#$$

*Note how the choice of interval ($[0,\pi]$ versus $[0,\pi/2]$) changes the inner product between functions.

[Proof of the Cauchy-Schwarz inequality:] $\left(|\langle v, w\rangle| \le \|v\| \, \|w\| \right)$.

If $y \neq 0$, consider

$$\|x - cy\|^2 = \langle x - cy, x - cy \rangle$$

$$= \langle x, x\rangle - c\langle x, y\rangle - c\langle y, x\rangle + c^2\langle y, y\rangle$$

symmetry

$$= \|x\|^2 - 2c\langle x, y\rangle + c^2\|y\|^2$$

Taking $c = \frac{\langle x, y\rangle}{\|y\|^2}$, then we have

$$0 \le \|x - cy\|^2 = \|x\|^2 - 2\frac{\langle x, y\rangle}{\|y\|^2}\langle x, y\rangle + \frac{\langle x, y\rangle^2}{\|y\|^4}\|y\|^2.$$

$$= \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2}$$

Then $\frac{\langle x, y\rangle^2}{\|y\|^2} \le \|x\|^2$, it implies $|\langle x, y\rangle| \le \|x\| \, \|y\|$.

When does equality hold in the Cauchy-Schwarz inequality?

$x = cy$ for some constant c.

Definition: Let V be an inner product space. We call $\mathbf{x}, \mathbf{y} \in V$ are orthogonal (perpendicular) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

In Example 1, we saw that

$$\int_0^\pi \cos(x)\sin(x)dx = 0$$

Thus, we say $f(x) = \cos(x)$ and $g(x) = \sin(x)$ are orthogonal to each other on the interval $[0, \pi]$.

A vector $\mathbf{w}$ in $\mathbb{R}^n$ is orthogonal to vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ if and only if $\mathbf{w}$ is in the cokernel of the matrix $A = [\mathbf{v}_1, \dots, \mathbf{v}_k]$ (i.e. in the kernel of A^T).

To see this: $\langle w, v_1 \rangle = 0, \dots, \langle w, v_k \rangle = 0$

So, $v_1^T w = 0, \dots, v_k^T w = 0$.

$$A^T w = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$ Then $w \in \ker(A^T) = \operatorname{coker}(A)$.

Example 2. Suppose we want to find a vector $\mathbf{w}$ that is orthogonal to $\mathbf{v}_1 = (1, 1, 1)^T$ and $\mathbf{v}_2 = (2, 1, -1)^T$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\xrightarrow{② - 2①} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\vec{w} = \begin{pmatrix} 2w_3 \\ -3w_3 \\ w_3 \end{pmatrix}, \quad w_3 \in \mathbb{R}.$$

$w_3 = 1$.

$$\vec{w} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}. \#$$

§ The Triangle Inequality

> Let V be an inner product space. Then the following **Triangle Inequality** holds:
> $$\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\| \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

[Proof:]

$$\|v+w\|^2 = \langle v+w, v+w\rangle$$
$$= \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle,$$
$$= \|v\|^2 + 2\langle v, w\rangle + \|w\|^2$$
$$\le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$$
$$= (\|v\| + \|w\|)^2.$$

By Cauchy–Schwarz inequality.

$\lceil (a+b)^2 = a^2 + 2ab + b^2 \rfloor$

Then $\|v+w\| \le \|v\| + \|w\|$.

* The equality holds in the triangle inequality iff $\cancel{\mathbf{x}}\ v = c\mathbf{w}$ and $c \ge 0$.

3.3 Norms

Recall some properties satisfied by the norm $\|\mathbf{v}\|$ on an inner product space, where $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$; (induced by the inner product)

(1)

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

$$\left(\begin{aligned} \|cv\|^2 &= \langle cv, cv \rangle \\ &= c^2 \langle v, v \rangle \\ &= c^2 \|v\|^2 \end{aligned} \right.$$

Then $\|cv\| = |c|\|v\|$. #.)

(2) $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$.

(3) the triangle inequality:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

There are other natural measures of the "size" of a vector that satisfy these same three conditions, but that cannot be defined in terms of an inner product.

For example,

(1) on $\mathbb{R}^n$ we can measure the size of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ by the sum of absolute values of its entries:

$$\sum_{i=1}^{n} |v_i|$$

(satisfies (1)–(3), will check it later)

(2) On $C^0([a, b])$, we can measure the size of a continuous function f by the integral of its absolute value:

$$\int_a^b |f(x)| dx$$

(satisfies (1)–(3), will check it later).

Neither of these quantities can be defined in terms of an inner product; but they are still useful notions of the "size of vectors".

Definition: A **norm** on a vector space V assigns a non-negative real number $\|\mathbf{v}\|$ to each vector $\mathbf{v} \in V$, such that for every $\mathbf{v},\ \mathbf{w} \in V$ and $c \in \mathbb{R}$, the following axioms holds:

(1) Positivity: $\|\mathbf{v}\| \geq 0$; $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

(2) Homogeneity: $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

(3) Triangle inequality: $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Remark: We checked that when "a norm is induced from an inner product, these three conditions are satisfied automatically". But in general, a norm need not arise from an inner product on V.

1. p **norm on** $\mathbb{R}^n$**.** We've already seen several on $\mathbb{R}^n$ and $C^0([a,b])$.

Example. We have seen that $\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the norm induced from the dot product on $\mathbb{R}^n$.

2 norm

1 norm.

Example. $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$. Let's check the three conditions to make sure this is a valid norm.

(1) $\|x\| \geq 0$; $\|x\| = 0$ implies $|x_i| = 0$, $i = 1, \dots, n$.
implies $x = 0$. and vice versa.

(2) $\|cx\| = \sum_{i=1}^n |cx_i| = |c| \left(\sum_{i=1}^n |x_i|\right) = |c|\,\|x\|$.

(3) $\|x+y\| = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + |y_i| \leq \left(\sum_{i=1}^n |x_i|\right) + \left(\sum_{i=1}^n |y_i|\right)$
$= \|x\| + \|y\|$.

Thus, $\|x\| = \sum_{i=1}^n |x_i|$ defines a norm on $\mathbb{R}^n$. #