Lecture 18: Quick review from previous lecture

- **Triangle Inequality** holds:

$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

- **Definition:** A **norm** on a vector space V assigns a non-negative real number $\|\mathbf{v}\|$ to each vector $\mathbf{v} \in V$, such that for every $\mathbf{v}$, $\mathbf{w} \in V$ and $c \in \mathbb{R}$, the following axioms holds:

  (1) Positivity: $\|\mathbf{v}\| \geq 0$; $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

  (2) Homogeneity: $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

  (3) Triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

- We have saw the following norm on $\mathbb{R}^n$:
  - (2 norm) $\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the norm induced from the dot product on $\mathbb{R}^n$.
  - (1 norm) $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$

Today we will discuss Norms.

Jesse(TA): No office hours on March 17th (Tues.) and March 19th.
Instead, March 24 (Tues.) and March 26 (Thur.) from 8:05 am to 12:05 pm

- Midterm 1 solutions are on Canvas now.

There is a generalization of these norms:

If $p \geq 1$, we define:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

This is often called the p **norm**.

Example. (1) Compute the 3 norm of $\mathbf{x} = (2, -1, 3)^T$:

$$\|x\|_3 = \left(2^3 + |-1|^3 + 3^3 \right)^{1/3} = \left(8 + 1 + 27 \right)^{1/3} = (36)^{1/3}.$$

(2) Compute the p norm of $\mathbf{x} = (1, \ldots, 1)^T$ in $\mathbb{R}^n$:

$$\|x\|_p = \big(\underbrace{1^p + \cdots + 1^p}_{n \text{ terms}} \big)^{1/p} = n^{1/p}.$$

2. L^p norm on $C^0[a,b]$.

We define the $\cancel{p}$ L^p norms on $C^0([a,b])$, for $p \geq 1$:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

Check the three conditions to make sure this is a norm:

1. Positivity: $\|f\|_p = 0 \iff f = 0$.
2. Homogeneity: $\|cf\|_p = |c| \, \|f\|_p$

} Exercise.

3. triangular inequality is tricky, we will skip it.

Example. Compute the $3/2$ norm of $f(x) = x^2$ on $[0,1]$.

$$\|f\|_{3/2} = \left(\int_0^1 |f|^{3/2} dx \right)^{2/3} = \left(\int_0^1 |x|^3 dx \right)^{2/3}.$$

$$= \left(\int_0^1 x^3 dx \right)^{2/3}.$$

$$= \left(\frac{x^4}{4} \Big|_0^1 \right)^{2/3} = \left(\frac{1}{4} \right)^{2/3}$$

$$= 2^{-4/3}$$

3. $p = \infty$.

when $p = \infty$, we define ∞ norm on $\mathbb{R}^n$ by:

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

Example. If $\mathbf{x} = (1, -5, 3)^T$, then $\|\mathbf{x}\|_\infty =$ 5.

Similarly,

when $p = \infty$, on $C^0([a,b])$ we define

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

Example. If $f(x) = -x^3$, then on $[-1, 1]$, $\|f\|_\infty = \max_{x \in [-1,1]} |f| = \max_{x \in [-1,1]} |-x^3|$

$= \max_{x \in [-1,1]} |x^3|$

$= 1$. #

3. Distance. Every norms defines a **distance** between vector space elements, that is,

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

It satisfies

1. Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$
2. Positivity: $d(\mathbf{v}, \mathbf{w}) = 0$ iff $\mathbf{v} = \mathbf{w}$.
3. Triangle inequality: $d(\mathbf{v}, \mathbf{w}) \le d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$

Recall $M_{m\times n} = \{ m\times n \text{ matrix} \}$

4. Matrix Norms.

If $\|\mathbf{v}\|$ is any norm on $\mathbb{R}^n$, it induces a natural norm on $\boxed{\mathcal{M}_{n\times n}}$, the vector space of n-by-n matrices.

The **matrix norm** (with respect to the norm $\|\cdot\|$ on $\mathbb{R}^n$) is defined as follows. If A is any n-by-n matrix, then

$$\|A\| = \max\{\|A\mathbf{u}\| : \|\mathbf{u}\| = 1\}$$

Proof. To show it is a norm, see pages 153–154 in the book for the proof. □

* In other words, $\|A\|$ is the maximum amount that A can change the norm of a unit vector $\mathbf{u}$ (one with $\|\mathbf{u}\| = 1$) when we apply A to $\mathbf{u}$.

* The book calls $\|A\|$ the **natural matrix norm** associated to the vector norm $\|\mathbf{v}\|$. It is also often called the **operator norm** of A.

Fact:

(1) $\|A\mathbf{v}\| \le \|A\|\|\mathbf{v}\|$, for all $n\times n$ matrices A, $\mathbf{v}\in\mathbb{R}^n$,

(2) $\|AB\| \le \|A\|\|B\|$, for all $n\times n$ matrices A and B.

[To see these:]

$\left\|\frac{u}{\|u\|}\right\| = \frac{1}{\|u\|}\|u\| = 1.$

(1) $u\neq 0$, unit vector $w=\frac{u}{\|u\|}$, $\|w\|=1$.

$$\|Aw\| = \left\|A\frac{u}{\|u\|}\right\| \le \|A\|. \Rightarrow \frac{\|Au\|}{\|u\|} \le \|A\|$$

$$\Rightarrow \|Au\| \le \|A\|\,\|u\|. \quad \#$$

(2) $\|u\|=1$.

$$\|ABu\| = \|A(Bu)\| \overset{(1)}{\le} \|A\|\,\|Bu\| \overset{(1)}{\le} \|A\|\,\|B\|\underbrace{\|u\|}_{=1} = \|A\|\|B\|.$$

$$\underbrace{\|AB\|}_{=} \max\{\|ABu\| ; \|u\|=1\} \le \|A\|\,\|B\|. \quad \#$$

The 2^{nd} inequality implies that

Fact: If A a square matrix, then

$$\|A^k\| \le \|A\|^k.$$

3.4 Positive Definite Matrices

We have seen that the following inner products on $\mathbb{R}^n$:

- standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n = (x_1, \cdots, x_n) \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$
  $= x^T I_n y.$
- For $c_1 > 0, \cdots, c_n > 0$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = c_1 x_1 y_1 + \ldots + c_n x_n y_n$$

$= x^T D y$, $D = \mathrm{diag}(c_1, \ldots, c_n)$

- For nonsingular matrix A,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T (A^T A) \mathbf{y}$$

  is also an inner product

They all have been of the following form:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$$

for some symmetric matrix K.

Q: Are there any other types of inner products on $\mathbb{R}^n$?

Ans: In fact, all inner products on $\mathbb{R}^n$ are of the form $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$, for some symmetric matrix K.

However, it is not true that any symmetric matrix K defines an inner product! Only a special type of matrix K can do this.

Recall: inner product.

1. Bilinearity
2. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
3. Positivity: $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \Leftrightarrow x = 0.$