Lecture 19: Quick review from previous lecture

- If $p \geq 1$, we define:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

  This is often called the p **norm**.

- We define the L^p norms on $C^0([a,b])$, for $p \geq 1$:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

- If A is any n-by-n matrix, then

$$\|A\| = \max\{\|A\mathbf{u}\| : \|\mathbf{u}\| = 1\}$$

Today we will discuss Positive definite matrix.

Jesse(TA): No office hours on March 17th (Tues.) and March 19th.
Instead, March 24 (Tues.) and March 26 (Thur.) from 8:05 am to 12:05 pm

- Quiz 4 (covers sec. 2.5, 3.1, 3.2) will take place in the beginning of the class on **Friday** 3/20

3.4 - 3.5 Positive Definite Matrices

Definition: An $n \times n$ matrix K is called **positive definite** if it is symmetric, $K^T = K$, and satisfies the positivity condition

$$\mathbf{x}^T K \mathbf{x} > 0 \qquad \text{for all } \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n.$$

We write $K > 0$ to mean that K is positive definite matrix.

Fact: Every inner product on $\mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where K is a **symmetric, positive definite** $n \times n$ matrix.

[To see it is an inner product:]

1 Bilinearity : Exercise

2 Symmetry : $\langle x, y \rangle = \langle y, x \rangle$.

$$\langle x, y \rangle = \overbrace{x^T K y}^{\text{scalar}} = (x^T K y)^T$$

$$= y^T K^T x$$

$$= y^T K x = \langle y, x \rangle.$$

#

3 positivity :

$$\langle x, x \rangle = x^T K x > 0, \ x \neq 0 \text{ since } K \text{ is positive definite}$$

$$\langle x, x \rangle = 0 \iff x^T K x = 0$$

$$\iff x = 0 \quad \text{due to } K \text{ is positive definite}$$

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Note: for a matrix A, the function

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

is called a **quadratic form**. "quadratic form is called "positive definite" if $q(x) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$

Warning: positive definite matrices may have negative entries, while entries with all positive entries may not always be positive definite.

Example.

$$K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$$

is positive definite, but it has negative entries.

$$q(x) = x^T K x = (x_1, x_2)\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 4x_1^2 - 4x_1x_2 + 3x_2^2$$

$$= (4x_1^2 - 4x_1x_2 + x_2^2) + 3x_2^2 - x_2^2$$

$$= (2x_1 - x_2)^2 + 2x_2^2 \geq 0.$$

$q(x) = 0 \iff 2x_1 = x_2$, $x_2 = 0 \iff x_1 = x_2 = 0 \iff x = (0,0)^T$.

So $K > 0$ #

Example.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

is Not positive definite

$$q(x) = x^T A x = x_1^2 + 6x_2x_1 + 2x_2^2$$

Taking $x_0 = (1, -1)^T$, $q(x_0) = -3 < 0$

So A is NOT positive definite

How can we tell if a matrix K is positive definite? We obviously can't evaluate $\mathbf{x}^T K\mathbf{x}$ for all vectors $\mathbf{x}$!

§ The positive definite 2×2 matrices.

Take any symmetric 2-by-2 matrix A:

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$q(x) = (x_1, x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boxed{a} x_1^2 + 2bx_1x_2 + cx_2^2.$$

(1) $a \le 0$, then A is NOT positive definite.

$\ulcorner$ $A = \begin{pmatrix} -3 & b \\ b & c \end{pmatrix}$. $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q(e_1) = e_1^T A e_1 = -3 < 0$. $\lrcorner$

Thus, we suppose $\boxed{a > 0}$.

(2) Complete the square:

$$q(x) = \left(\sqrt{a}\, x_1 + \frac{b}{\sqrt{a}} x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2.$$

$$q(x) > 0 \quad \text{if} \quad \boxed{c - \frac{b^2}{a} > 0} \left(\boxed{\underbrace{ac - b^2}_{\det A} > 0}\right).$$

Fact: $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite if and only if

$$a > 0 \quad \text{and} \quad ac - b^2 > 0$$

§ The positive definite $n \times n$ matrices.

Is there a simple characterization for positive definite matrices of any size?

Recall: Any regular symmetric matrix A can be factored in the form

$A \xrightarrow{\text{Gaussian}} U = \begin{pmatrix} u_{11} & & \\ & \ddots & \\ 0 & & u_{nn} \end{pmatrix}$, $u_{11} \neq 0, \ldots, u_{nn} \neq 0$

$$A = LDL^T,$$

where L is lower unitriangular and D is diagonal.

(*This factorization is computed via Gaussian elimination.)

1. 2×2 **symmetric matrix** Suppose $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

$a > 0$, By Gaussian - Elimination,

$A \xrightarrow{\textcircled{2} - \frac{b}{a}\textcircled{1}} U = \begin{pmatrix} a & b \\ 0 & c - \frac{b^2}{a} \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}$.

$$A = LU = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c - \frac{b^2}{a} \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & c - \frac{b^2}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

$$= \quad L \qquad \overset{\text{"}}{D} \qquad L^T$$

We have seen that

A is positive definite $\iff a > 0,\ c - \frac{b^2}{a} > 0$.

$\iff$ all diagonal entries of D are positive.

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2. Any size symmetric matrix

It turns out that this same criterion characterizes positive definite matrices of any size.

Fact: An n-by-n matrix A is positive definite if and only if it is:
(a) symmetric;
(b) regular, hence $A = LDL^T$; and
(c) D has all positive diagonal entries, i.e. A has positive pivots.

Fact: If a matrix is positive definite, then it is nonsingular.

Example. Suppose A is the symmetric matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Gaussian Elimination.

$$\xrightarrow{②-①} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix} \to \text{zero pivot (A is NOT regular)}$$

Then A is NOT positive definite.

Example. Suppose A is the symmetric matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 5 \end{pmatrix}$$

$$A \xrightarrow[\textcircled{3}-2\textcircled{1}]{\textcircled{2}+\textcircled{1}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{\textcircled{3}+\textcircled{2}} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix}}_{}$$

U

negative.

So A is NOT positive definite.

Example. Suppose A is the symmetric matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 7 \end{pmatrix}$$

$$\longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

Thus, $\underline{A > 0}$ #

Definition: If a matrix A satisfies $\mathbf{x}^T A\mathbf{x} \geq 0$ for all vectors $\mathbf{x}$, it is called **positive semidefinite**.

Remark: Every positive definite matrix is also positive semidefinite; but the converse is not true, since a positive semidefinite matrix A might have $\mathbf{x}^T A\mathbf{x} = 0$ for $\mathbf{x} \neq \mathbf{0}$.

Example. The matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite, but not positive definite.

$q(x) = \underset{x^TAx}{\wedge} = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0.$

Thus A is positive semidefinite.

$q(x) = 0 \iff x_1 = x_2.$

So A is <u>NOT</u> positive definite.

Definitions:

- a matrix A is **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- Similarly, a matrix A is **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x}$.
- If a matrix is neither positive or negative semidefinite, it is called **indefinite**. This means that there are vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{y}^T A\mathbf{y} < 0$.

*Only "positive definite" matrices define inner products, via $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}$.