Lecture 2: Quick review from previous lecture

- Gaussian elimination to solve a linear system $A\mathbf{x} = \mathbf{b}$.
- $I_n$ is the $n$-by-$n$ identity matrix, defined by:

$$I_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}_{n \times n}$$

or $I_n = \text{diag}(1, \cdots, 1)$.

- The first problem set has been posted on Canvas. It is due next Friday, 1/31, at the end of class.
- There will be a quiz in class on Wednesday (1/29).
1.3 Gaussian Elimination

• Suppose $E$ is a 3-by-3 elementary matrix that adds 7 times the 1\textsuperscript{st} row to the 3\textsuperscript{rd} row. Then:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

• How to UNDO the effect of this row operation?

Subtracting $7\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ from $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$:

We denote it by $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}$. 

$EE^{-1} = E^{-1}E = I_3$.

• When performing Gaussian elimination, when we reach the $j^{th}$ row, element $(j, j)$ of the new augmented matrix is called the pivot for that row.

**Example:** We look at the example:

$$\begin{cases} x + 2y + 2z = 2 \\ 2x + 10y = 1 \\ 4x + y + 4z = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 10 & 0 & 1 \\ 4 & 1 & 4 & 0 \end{pmatrix}$$

1st pivot is element $(1, 1)$, that is, 1.

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 16 & \textcolor{red}{-4} & \textcolor{red}{-3} \\ 0 & -7 & -4 & \textcolor{red}{-8} \end{pmatrix}$$

2nd pivot is $(2, 2)$, that is, 6.
\[ \frac{3}{4} \begin{pmatrix} 1 & 2 & 2 \\ 6 & -4 & -3 \\ 0 & 6 & \frac{2}{3} \end{pmatrix} \]

3rd pivot is \( 3, 3 \), that is, \( \frac{-26}{3} \).

**Exercise:** Find \( x, y, z \) by using back-substitution.
✓ If at any point in the process one of the pivots is 0, then we are stuck! We can’t use a row with a zero pivot to eliminate the entries beneath that pivot.

**Example:** Suppose we are solving a 4-by-4 system and after using the first row to eliminate entries (2, 1), (3, 1), and (4, 1), we have the following matrix:

$$
\begin{pmatrix}
5 & 2 & 3 & 5 & | & 2 \\
0 & \underline{2} & 6 & 9 \\
0 & 1 & 3 & 8 & | & 3 \\
0 & 2 & 5 & 1 & | & 8
\end{pmatrix}
$$

- How to fix this? **We will permute row 2 with other row** (will be discussed later).

- If a matrix $A$ has all non-zero pivots, it is called **regular**. That is, regular matrices are those for which Gaussian elimination can be performed without switching the order of rows.
For any regular matrix $A$, we can multiply it on the left by a sequence of elementary matrices $E_1, \ldots, E_m$, so that the product is an upper triangular matrix $U$:

$$E_m E_{m-1} \cdots E_1 A = U$$

\[ \text{upper triangular matrix.} \]

\[ \text{Some observation:} \]

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, associated to row 2 + a row 1

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$, \quad \therefore \quad \text{row} 2 + b \text{ row} 1

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$, \quad \therefore \quad \text{row} 3 + c \text{ row} 1

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-c \end{pmatrix}$.

Then $E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1-bc \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-c \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b - c & 0 & 1 \end{pmatrix} \text{ lower triangular (zero above main diagonal)}$$

We now can see that $E_1^{-1} E_2^{-1} \cdots E_m^{-1}$ has the form

$$L = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ x & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \end{bmatrix}$$
Then

\[
E_m \ldots E_1 A = LU
\]

\[
( E_1^T \ldots E_{m-1}^T \ E_m^T ) \bar{E}_m \ldots \bar{E}_1 A = (E_1^T \ldots E_{m-1}^T \ E_m^T) \bar{U}
\]

So, \[ A = I \bar{A} = (E_1^T \ldots E_{m-1}^T \ E_m^T) \bar{U}. \]

denoted by \( \bar{L} \)

**Facts:**

(1) We have shown that any regular matrix \( A \) can be factored as \( A = LU \), where \( U \) is upper triangular and \( L \) is lower triangular.

Furthermore, \( L \) has 1’s on its main diagonal, and \( U \) has non-zero elements on its main diagonal (the pivots of \( A \)).

(2) \( L, \tilde{L} \) are \( n \times n \) lower triangular matrices, so is \( L\tilde{L} \).

(3) \( U, \tilde{U} \) are \( n \times n \) upper triangular matrices, so is \( U\tilde{U} \).