Lecture 2: Quick review from previous lecture

- Gaussian elimination to solve a linear system $A\mathbf{x} = \mathbf{b}$.
- I_n is the n-by-n *identity matrix*, defined by:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

  or $I_n = \mathrm{diag}(1, \cdots, 1)$.

- The first problem set has been posted on Canvas. It is due next Friday, 1/31 , at the end of class.
- There will be a quiz in class on Wednesday (1/29).

1.3 Gaussian Elimination

- Suppose E is a 3-by-3 elementary matrix that adds 7 times the 1^{st} row to the 3^{rd} row. Then:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

- How to UNDO the effect of this row operation?

substracting 7 ① from ③:

We denote it by $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}$.

$EE^{-1} = E^{-1}E = I_3$.

- When performing Gaussian elimination, when we reach the j^{th} row, element (j, j) of the new augmented matrix is called the **pivot** for that row.

Example: We look at the example:

$$\begin{cases} x + 2y + 2z &= 2 \\ 2x + 10y &= 1 \\ 4x + y + 4z &= 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 2 & 2 \\ 2 & 10 & 0 & 1 \\ 4 & 1 & 4 & 0 \end{array}\right)$$

1st pivot is element (1,1), that is, 1.

$$\xrightarrow[③-4①]{②-2①} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & \boxed{6} & -4 & -3 \\ 0 & -7 & -4 & -8 \end{array}\right)$$

2nd pivot is (2,2), that is, 6.

[Example Continue]

$$\xrightarrow{③ + \frac{7}{6}②} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 6 & -4 & -3 \\ 0 & 0 & \boxed{\frac{-26}{3}} & \frac{-23}{2} \end{array}\right).$$

3rd pivot " (3,3), that is, $\frac{-26}{3}$.

[Exercise]: find x, y, z by using "back-substitution".

✓ If at any point in the process one of the pivots is 0, then we are stuck! We can't use a row with a zero pivot to eliminate the entries beneath that pivot.

Example: Suppose we are solving a 4-by-4 system and after using the first row to eliminate entries $(2,1)$, $(3,1)$, and $(4,1)$, we have the following matrix:

$$\left(\begin{array}{cccc|c} 5 & 2 & 3 & 5 & 2 \\ 0 & 0 & 2 & 6 & 9 \\ 0 & 1 & 3 & 8 & 3 \\ 0 & 2 & 5 & 1 & 8 \end{array}\right)$$

- How to fix this? We will permute row 2 with other row (will discussed later).

- If a matrix A has all non-zero pivots, it is called **regular**. That is, regular matrices are those for which Gaussian elimination can be performed without switching the order of rows.

For any regular matrix A, we can multiply it on the left by a sequence of elementary matrices $E_1, \ldots, E_m$, so that the product is an upper triangular matrix U:

$$E_m E_{m-1} \cdots E_1 A = \underline{U}$$ — upper triangular matrix.

* Some observation:

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, associated to row 2 + a row 1

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$, " row 3 + b row 1

$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$, " 3 + c 2.

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix}$.

Then $E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{pmatrix}$$

lower triangular (zero above main diagonal)

We now can see that

$E_1^{-1} E_2^{-1} \cdots E_m^{-1}$ has the form

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \times & 1 & & \\ \times & & \ddots & 0 \\ \vdots & & & \\ \times & \cdots & \times & 1 \end{bmatrix}$$

Then $$E_m \cdots E_1 A = U$$

$$(E_1^{-1} \cdots E_{m-1}^{-1} E_m^{-1}) E_m \cdots E_1 A = (E_1^{-1} \cdots E_{m-1}^{-1} E_m^{-1}) U$$

So, $$A = IA = \underbrace{(E_1^{-1} \cdots E_{m-1}^{-1} E_m^{-1})}_{\text{denoted by } L} U.$$

Facts:

(1) We have shown that any regular matrix A can be factored as $A = LU$, where U is upper triangular and L is lower triangular.
Furthermore, L has 1's on its main diagonal, and U has non-zero elements on its main diagonal (the pivots of A).

(2) $L, \tilde{L}$ are $n \times n$ lower triangular matrices, so is $L\tilde{L}$.

(3) $U, \tilde{U}$ are $n \times n$ upper triangular matrices, so is $U\tilde{U}$.