Lecture 20: Quick review from previous lecture

- An $n \times n$ matrix K is called **positive definite** if
  - it is symmetric, $K^T = K$
  - satisfies the positivity condition

$q(x) = x^T K x$ is called "quadratic form".

$$\mathbf{x}^T K \mathbf{x} > 0 \qquad \text{for all } \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n.$$

We write $K > 0$ to mean that K is positive definite matrix.

$*$ K is positive semi definite $x^T K x \geq 0$, for all x.

- Identify 2×2 positive definite matrix:

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite if and only if

$$a > 0 \;\text{ and }\; \overbrace{ac - b^2}^{\text{"} \det A} > 0$$

- Identify any $n \times n$ positive definite matrix:
  An n-by-n matrix A is positive definite if and only if it is:
  (a) symmetric;
  (b) regular, hence $A = LDL^T$; and
  (c) D has all positive diagonal entries, i.e. A has positive pivots.

$A \xrightarrow{\text{Gaussian}} U = \begin{pmatrix} u_{11} & * & * \\ & \ddots & * \\ 0 & & u_{nn} \end{pmatrix}$, $u_{11} \neq 0, \ldots, u_{nn} \neq 0$. , $A = LU = L\underbrace{DL^T}_{D}$

Today we will continue our discuss on Positive definite matrix.

- I will hold extra office hour this Thursday (3/19) from 8:30-9:30am
  Zoom meeting ID: 904-508-509
- Jesse(TA): will office hours as usual this week.
  His Zoom meeting ID: 707-312-921

(Those meeting IDs can be found in our Canvas course website.)

- Quiz 4 is Canceled; No quizzes for the remainder of the semester.

HW 14 % ; Quiz : 4 %

§ Constructing positive definite or positive semidefinite matrices

Fact: Suppose A is any $m \times n$ matrix. Then $K = A^T A$ is positive semidefinite.

$x^T K x \geq 0$

[To see this:]

$$x^T K x = x^T (A^T A) x$$
$$= (x^T A^T) A x, \quad z = Ax, \quad z^T = (Ax)^T = x^T A^T.$$
$$= z^T z$$
$$= \|z\|^2 \geq 0.$$

Thus, K is positive semidefinite.

$$x^T K x = 0 \iff x = 0 \quad (\text{if } K > 0)$$
$$z = Ax = 0$$

Fact: $K = A^T A$ is positive definite when the rank of A is n (in particular, we must have $n \leq m$); or equivalently, the columns of A are linearly independent.

$A = [v_1 \cdots v_n]$, v_i: i^{th} column of A, $1 \leq i \leq n$.

$x^T K x = 0$,

$$x^T A^T A x = 0.$$
$$(Ax)^T Ax = 0$$
$$\|Ax\|^2 = 0$$
$$Ax = 0.$$
$$[v_1 \cdots v_n] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$
$$x_1 v_1 + \cdots + x_n v_n = 0.$$

Since $v_1, \ldots, v_n$ are linearly independent,

$x_1 = 0, \ldots, x_n = 0$ $(x = 0)$.

Thus. $K > 0$.

*If we write $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, then entry (i, j) of $A^T A$ is $\mathbf{v}_i^T \mathbf{v}_j = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. That is, $A^T A$ is the matrix of all inner products between the columns of A.

This is called a **Gram matrix**. More generally, if V is any inner product space, then the Gram matrix for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the matrix K given by

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \cdots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix}_{n \times n} = A^T A, \text{ where } A = [v_1 \cdots v_n].$$

Fact: (1) In $\mathbb{R}^n$, Gram matrices are always positive semidefinite;
(2) they are positive definite precisely when the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Example.

(1) $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}_{3 \times 2}$ (columns v_1, v_2) then the Gram matrix for A is:

① $\{v_1, v_2\}$ are linearly independent. So $A^TA > 0$.

② $6 > 0$, $\det\begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix} = 6 - 4 = 2 > 0$. So $A^TA > 0$.

$$K = A^T A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}_{2 \times 2}$$

(2) $B = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 1 & -2 \end{pmatrix}$ (columns v_1, v_2) then the Gram matrix for B is:

$$B^T B = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -12 \\ -12 & 24 \end{pmatrix}$$

$v_2 = -2v_1$, $\{v_1, v_2\}$ linearly dependent, $B^TB \geq 0$, but not $B^TB > 0$.

More generally, let's take any m-by-m positive definite matrix C. Then

$$\boxed{\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T C \mathbf{y}}$$

defines a valid inner product on $\mathbb{R}^m$.

Then the Gram matrix of $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix}$$

with respect to inner product $\boxed{\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \mathbf{v}_i^T C \mathbf{v}_j.}$

If $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is any m-by-n matrix, then

$$K = A^T \boxed{C} A.$$

* Prevous page is the "special case" with $C = I_n$.

Fact: Let C be a positive definite matrix.
(1) $K=$ $A^T C A$ is positive semidefinite,
(2) $K=$ $A^T C A$ is positive definite if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent (i.e. $\ker A = \{\mathbf{0}\}$).

(1) $x^T K x = x^T (A^T C A) x$
$= z^T C z$ $\quad z = Ax$

$\begin{cases} > 0 & \text{if } z = Ax \neq 0. \\ = 0 & \text{if } z = Ax = 0. \end{cases}$ due to $C > 0$.

$x^T K x \geq 0$. Thus, we can get $K \geq 0$.

(2) $x^T K x = 0 \iff z = Ax = 0$.

$\iff [v_1 \cdots v_n] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad (x_1 v_1 + \cdots + x_n v_n = 0)$.

since $\{v_1, \ldots, v_n\}$ l. independent

$\iff x_1 = \cdots = x_n = 0, \quad (x = 0)$

Thus, $K > 0$. #

Fact: Let $K = A^T C A$, where $A \in M_{m\times n}$ and C is a $m \times m$ positive definite matrix. Then

$$\ker K = \ker A$$

(1) $x \in \text{Ker } K$. We need to show $x \in \text{Ker } A$.

$x \in \text{Ker } K$. Then $Kx = 0$.

$x^T K x = 0$

$\Rightarrow x^T A^T C A x = 0$ $\quad z = Ax$

$\Rightarrow z^T C z = 0$

$\Rightarrow \underset{\overset{\shortparallel}{Ax}}{z} = 0$ since $C > 0$.

$\Rightarrow x \in \text{Ker } A$. #

(2) $x \in \text{Ker } A$. We need to show $x \in \text{Ker } K$.

$x \in \text{Ker } A$.

$Ax = 0$.

$Kx = (A^T C A)x = A^T C \overbrace{(Ax)}^{0} = A^T C 0 = 0$.

So, $x \in \text{Ker } K$. #

§ To find the symmetric matrix A from the quadratic form

Suppose $A = (a_{ij})$ is an m-by-n matrix.

The formula for $\mathbf{x}^T A\mathbf{y}$ is:

$$\mathbf{x}^T A\mathbf{y} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j$$

In particular, with a quadratic form $\mathbf{x}^T A\mathbf{x}$ defined by the symmetric matrix $A = (a_{ij})$, $a_{ij} = a_{ji}$ (square, of size n-by-n), we have

$$\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

How do we go backwards to find the symmetric A from $\mathbf{x}^T A\mathbf{x}$?

Example.

1. In 2 dimensions, suppose

$3x_1x_1$

$$\mathbf{x}^T A\mathbf{x} = 3x_1^2 - 4x_1x_2 + 7x_2^2.$$

$-4/2$

Then

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}_{2\times 2}$$

2. In 3 dimensions, suppose

$$\mathbf{x}^T K\mathbf{x} = x_1^2 + 4x_1x_2 - 2x_1x_3 + 6x_2^2 + 9x_3^2.$$

$4/2 = 2$

$-2/2 = -1$

Then

$$K = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix}$$

3×3