Lecture 21: Quick review from previous lecture

$K > 0$. positive definite
① $K = K^T$
② $x^T K x > 0$, for all $x \neq 0$.

$\langle v_i, v_j \rangle = v_i^T C v_j$, $C > 0$.

- The **Gram matrix** is

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \dots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix}_{n \times n}$$

  - In $\mathbb{R}^n$, Gram matrices are always positive semidefinite
  - they are positive definite precisely when the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

Today we will discuss "Orthogonal(Orthonormal) bases"

Lecture video can be found in Canvas "Media Gallery"

3.6 Complex Vector Spaces

To finish Chapter 3: let's briefly discuss complex vector spaces and complex inner products.

Recall that a **complex number** as an expression of the form

$$z = x + iy, \qquad x,\ y \in \mathbb{R}.$$

$i = \sqrt{-1}$. $i^2 = (\sqrt{-1})^2 = -1$.

The **complex conjugate** of $z = x + iy$ is

$$\overline{z} = x - iy.$$

Thus,

$$|z|^2 = z\overline{z}$$

- Everything we have done in Chapter 1 with $\mathbb{R}^n$ and real-valued scalars works in $\mathbb{C}^n$ with complex-valued scalars.
- In particular, Gaussian elimination works exactly the same way if the numbers are in $\mathbb{C}$ instead of $\mathbb{R}$.

$z = 1 + i$. $(1 + \sqrt{-1})$.

$\bar{z} = 1 - i$.

$|z|^2 = z\bar{z} = (1+i)(1-i)$

$= 1 - i + i - i^2 = 1 + 1 = 2$.

✓ $|z| = |1+i| = \sqrt{2}$. #

In general, $z = x + iy$.

$|z|^2 = x^2 + y^2$, $|z| = \sqrt{x^2 + y^2}$.

§ Inner product on $\mathbb{C}^n$:

$$\langle \mathbf{w}, \mathbf{z} \rangle = \sum_{i=1}^{n} w_i \overline{z_i}$$

- This way, $\langle \mathbf{z}, \mathbf{z} \rangle = \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2$, which is positive if $\mathbf{z} \neq 0$.
- Note that $\langle \mathbf{w}, \mathbf{z} \rangle$ is not symmetric; rather it is *conjugate-symmetric*:

$$\langle \mathbf{w}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}$$

- For c, $d \in \mathbb{C}$,

$$\langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle$$
$$\langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle = \bar{c}\langle \mathbf{u}, \mathbf{v} \rangle + \bar{d}\langle \mathbf{u}, \mathbf{w} \rangle$$

- $\|v\|^2 = \langle v, v \rangle \geq 0$; $\langle v, v \rangle = 0 \iff v = 0$.

EX: $v = (1+i, 2i, -3)^T$. Find $\|v\|$.

$$\|v\| = \sqrt{|1+i|^2 + |2i|^2 + |-3|^2}$$
$$= \sqrt{2 + 4 + 9}$$
$$= \sqrt{15}. \quad \#$$

$|x+iy|^2 = x^2 + y^2$

$|2i|^2 = 0^2 + 2^2 = 4.$

4 Orthogonality

4.1 Orthogonal and Orthonormal Bases

We've already seen that in an inner product space V, two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Fact: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero vectors that are "mutually orthogonal", meaning $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

[To see this]: Consider $c_1 v_1 + \cdots + c_n v_n = 0$.

We want to show $c_1 = 0, \ldots, c_n = 0$.

$\langle c_1 v_1 + \cdots + c_n v_n, v_1 \rangle = \langle 0, v_1 \rangle$.

$c_1 \langle v_1, v_1 \rangle + c_2 \cancel{\langle v_2, v_1 \rangle}^{0} + \cdots + c_n \cancel{\langle v_n, v_1 \rangle}^{0} = 0$.

$c_1 \langle v_1, v_1 \rangle = 0$

$c_1 \|v_1\|^2 = 0 \Rightarrow c_1 = 0$. Similarly, we can get $c_2 = 0, \ldots, c_n = 0$

Definition: Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero vectors that are mutually orthogonal. If additionally $\|\mathbf{v}_i\| = 1$, we say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **orthonormal**.

Definition:

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are mutually ① orthogonal vectors that are also a ② basis for V (so $\dim V = n$), we say they are an **orthogonal basis**.
- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are ① orthonormal and a ② basis for V, we say they are an **orthonormal basis**. $(\|v_1\| = \cdots = \|v_n\| = 1)$.

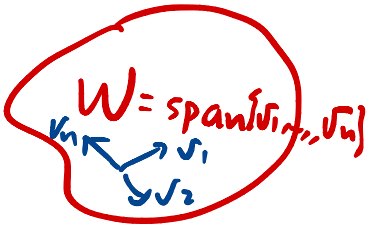
Example. In $\mathbb{R}^n$ equipped with the standard dot product, an orthonormal basis is the standard basis:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

① $\langle e_1, e_2 \rangle = 0, \ldots, \langle e_i, e_j \rangle = 0$, $i \neq j$, $1 \le i, j \le n$.

② $\|e_j\| = 1$, $1 \le j \le n$.

Fact: Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero, mutually orthogonal (resp. orthonormal) vectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal (resp. orthonormal) basis for their span $W = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.



$\{v_1, \ldots, v_n\}$ orthogonal (orthonormal) basis for W. #

Fact: Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is orthogonal basis. Then

$$\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$$

form an orthonormal basis.

$$\left\| \frac{v_1}{\|v_1\|} \right\| = \frac{1}{\|v_1\|} \|v_1\| = 1.$$

Example. Explain why vectors $\mathbf{x} = (1 \;\; 1 \;\; 0)^T$, $\mathbf{y} = (1 \;\; -1 \;\; 1)^T$, and $\mathbf{z} = (1 \;\; -1 \;\; -2)^T$ form an orthogonal basis in $\mathbb{R}^3$? Turn them into an orthonormal basis.

x, y, z are mutually orthogonal. $\xRightarrow{\text{Fact}}$ $\{x, y, z\}$ linearly independent

$$\langle x, y \rangle = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0.$$

$$\langle x, z \rangle = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\rangle = 0.$$

$$\langle y, z \rangle = 0.$$

Thus, $\{x, y, z\}$ is an orthogonal basis in $\mathbb{R}^3$.

$$\left\{ \frac{x}{\|x\|}, \frac{y}{\|y\|}, \frac{z}{\|z\|} \right\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

is an orthonormal basis for $\mathbb{R}^3$.

§ Computations in Orthogonal Bases

What are the advantages of orthogonal (orthonormal) bases?

It is simple to find the coordinates of a vector in the orthogonal (orthonormal) basis. However, in general this is not so easy.

Fact: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthogonal basis in any inner product space V, then for any vector $\mathbf{v} \in V$ we have

(1)
$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n,$$

where

$$\boxed{a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i\rangle}{\|\mathbf{v}_i\|^2},} \quad i = 1, \cdots, n.$$

Moreover, we have

(2)
$$\|\mathbf{v}\|^2 = a_1^2\|\mathbf{v}_1\|^2 + \cdots a_n^2\|\mathbf{v}_n\|^2 = \sum_{i=1}^{n}\left(\frac{\langle \mathbf{v}, \mathbf{v}_i\rangle}{\|\mathbf{v}_i\|}\right)^2.$$

[To see this:] (1) For any $v \in V$, we can write

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

$$\langle v, v_1\rangle = \langle a_1 v_1 + a_2 v_2 + \cdots + a_n v_n, v_1\rangle$$

$$= a_1\langle v_1, v_1\rangle + a_2\cancel{\langle v_2, v_1\rangle}^{0} + \cdots + a_n\cancel{\langle v_n, v_1\rangle}^{0}$$

Thus, $\langle v, v_1\rangle = a_1\|v_1\|^2$.

$$\Rightarrow a_1 = \frac{\langle v, v_1\rangle}{\|v_1\|^2}.$$

Similarly, we can get $a_2 = \dfrac{\langle v, v_2\rangle}{\|v_2\|^2}, \ldots,$ $a_n = \dfrac{\langle v, v_n\rangle}{\|v_n\|^2}$. #

* $(a_1, a_2, \ldots, a_n)$ is coordinate of vector v in basis $\{v_1, \ldots, v_n\}$.

(2) Exercise.

Example. Consider the orthogonal basis $\mathbf{x} = (1 \ \ 1 \ \ 0)^T$, $\mathbf{y} = (1 \ \ -1 \ \ 1)^T$, and $\mathbf{z} = (1 \ \ -1 \ \ -2)^T$ of $\mathbb{R}^3$. Write $\mathbf{v} = (1 \ \ 2 \ \ 3)^T$ as the linear combination of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

$$v = \frac{\langle v, x\rangle}{\|x\|^2} x + \frac{\langle v, y\rangle}{\|y\|^2} y + \frac{\langle v, z\rangle}{\|z\|^2} z.$$

$$\langle v, x\rangle = \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle = 1+2 = 3$$

$$\|x\|^2 = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle = 2.$$

$$\langle v, y\rangle = \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle = 1-2+3 = 2.$$

$$\|y\|^2 = \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle = 3.$$

$$\langle v, z\rangle = -7.$$

$$\|z\|^2 = 6.$$

Thus, $v = \frac{3}{2} x + \frac{2}{3} y - \frac{7}{6} z$ #

Example.

(1) The basis $1, x, x^2$ do NOT form an orthogonal basis. for $\mathcal{P}^{(2)}([0,1])$

$$\langle 1, x\rangle = \int_0^1 1\, x \, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2} \neq 0.$$

$$\langle x, x^2\rangle = \int_0^1 x \, x^2 \, dx = \int_0^1 x^3 \, dx = \frac{1}{4} \neq 0$$

$\vdots$

(2) ↓ continue Next Monday.

$$\boxed{p_1(x) = 1, \qquad p_2(x) = x - \frac{1}{2}, \qquad p_3(x) = x^2 - x + \frac{1}{6}.}$$

is an orthogonal basis of $\mathcal{P}^{(2)}([0,1])$.

Text (3) Write $p(x) = x^2 + x + 1$ in terms of the basis p_1, p_2, p_3 in (2).

(P. 190)