Lecture 22: Quick review from previous lecture

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero vectors that are **mutually orthogonal**, meaning $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$,

$$\text{mutually orthogonal} \Rightarrow \ \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly independent}$$

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for V and orthogonal (orthonormal), we call they are **orthogonal (orthonormal) basis**.

$\Downarrow$ $\|\mathbf{v}_i\| = 1, \ 1 \le i \le n.$

[Property:]

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an "orthogonal basis" in any inner product space V, then for any vector $\mathbf{v} \in V$ we have

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n,$$

where

$$\boxed{a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \ i = 1, \cdots, n.}$$

Today we will discuss examples for "Orthogonal(Orthonormal) bases" and then turn to the Gram-Schmidt process.

- Lecture will be recorded -

Lecture video can be found in Canvas "Media Gallery"

Example. Consider the orthogonal basis $\mathbf{x} = (1 \;\; 1 \;\; 0)^T$, $\mathbf{y} = (1 \;\; -1 \;\; 1)^T$, and $\mathbf{z} = (1 \;\; -1 \;\; -2)^T$ of $\mathbb{R}^3$. Write $\mathbf{v} = (1 \;\; 2 \;\; 3)^T$ as the linear combination of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

$$v = \frac{\langle v, x\rangle}{\|x\|^2} x + \frac{\langle v, y\rangle}{\|y\|^2} y + \frac{\langle v, z\rangle}{\|z\|^2} z.$$

$$\langle v, x\rangle = \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle = 1+2=3$$

$$\|x\|^2 = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle = 2.$$

$$\langle v, y\rangle = \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle = 1-2+3=2.$$

$$\|y\|^2 = \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle = 3.$$

$$\langle v, z\rangle = -7.$$

$$\|z\|^2 = 6.$$

Thus, $v = \frac{3}{2} x + \frac{2}{3} y - \frac{7}{6} z$ #

Example.

(1) The basis $1, x, x^2$ do NOT form an orthogonal basis. for $\mathcal{P}^{(2)}([0,1])$

$$\langle 1, x\rangle = \int_0^1 1\, x \, dx = \frac{1}{2}x^2\Big|_0^1 = 1/2 \neq 0.$$

$$\langle x, x^2\rangle = \int_0^1 x \, x^2 \, dx = \int_0^1 x^3 \, dx = \frac{1}{4} \neq 0$$

$\vdots$

$\uparrow 3/20$

$\downarrow 3/23$

(2)

$$p_1(x) = 1, \qquad p_2(x) = x - \frac{1}{2}, \qquad p_3(x) = x^2 - x + \frac{1}{6}.$$

is an orthogonal basis of $\mathcal{P}^{(2)}([0,1])$

(3) Write $p(x) = x^2 + x + 1$ in terms of the basis p_1, p_2, p_3 in (2).

(2) $$\langle p_1, p_2\rangle = \int_0^1 1\left(x - \tfrac{1}{2}\right) dx = \tfrac{1}{2}x^2 - \tfrac{1}{2}x\Big|_0^1 = 0.$$

$$\langle p_1, p_3\rangle = \int_0^1 1\left(x^2 - x + \tfrac{1}{6}\right) dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = \frac{2-3+1}{6} = 0.$$

$\langle P_2, P_3 \rangle = \int_0^1 (x-\frac{1}{2})(x^2-x+\frac{1}{6})\,dx = \cdots = 0.$

So, $\{P_1, P_2, P_3\}$ is mutually orthogonal,
by Fact, $\{P_1, P_2, P_3\}$ is linearly independent.
Thus, $\{P_1, P_2, P_3\}$ is a basis and moreover, it is "orthogonal" basis. #

(3). $P(x) = \underbrace{\frac{\langle P, P_1 \rangle}{\|P_1\|^2}}_{C_1} P_1 + \underbrace{\frac{\langle P, P_2 \rangle}{\|P_2\|^2}}_{C_2} P_2 + \underbrace{\frac{\langle P, P_3 \rangle}{\|P_3\|^2}}_{C_3} P_3$

① $\langle P, P_1 \rangle = \int_0^1 (x^2+x+1)\,1\,dx = \frac{1}{3}x^3+\frac{1}{2}x^2+x\Big|_0^1$
$= \frac{1}{3}+\frac{1}{2}+1$
$= \frac{2}{6}+\frac{3}{6}+\frac{6}{6}$
$= \boxed{\frac{11}{6}}.$

$\|P_1\|^2 = \langle P_1, P_1 \rangle = \int_0^1 1 = \boxed{1}$

$C_1 = \frac{11}{6} \Big/ 1 = 11/6$. #

② $\langle P, P_2 \rangle = \int_0^1 (x^2+x+1)(x-\frac{1}{2})\,dx = \_\_$

$\|P_2\|^2 = \int_0^1 (x-\frac{1}{2})^2\,dx = \frac{1}{12}.$

$C_2 = \frac{\langle P, P_2 \rangle}{\|P_2\|^2} = \boxed{2}.$

③ $\langle P, P_3 \rangle = \int_0^1 (x^2+x+1)(x^2-x+\frac{1}{6})\,dx$

$\|P_3\|^2 = 1/180$.

$C_3 = \boxed{1}$ Thus. $P = \frac{11}{6}P_1 + 2P_2 + P_3$ #

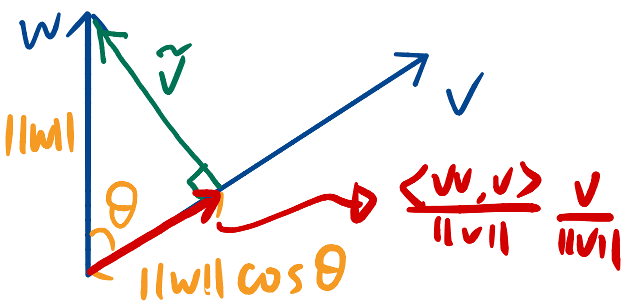
4.2 The Gram-Schmidt Process

Q: How can we construct the orthogonal (or orthonormal) bases?
This can be done by the algorithm, known as **the Gram-Schmidt process**.

§ Given 2 two vectors $\mathbf{v}, \mathbf{w}$

$* \ \langle w, v\rangle = \|w\| \|v\| \cos\theta.$

Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors, and $\mathbf{v} \neq \mathbf{0}$. How do we make up an vector orthogonal to $\mathbf{v}$ so that it forms an orthogonal set that spans the same subspace as $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$?



$$= \cancel{\|w\|}\left(\frac{\langle w, v\rangle}{\cancel{\|w\|}\|v\|}\right) = \frac{\langle w, v\rangle}{\|v\|}$$

$$\tilde{v} = w - \frac{\langle w, v\rangle}{\|v\|^2} v.$$

Thus, $\tilde{v}$ is orthogonal to v:

$$\langle \tilde{v}, v\rangle = \left\langle w - \frac{\langle w, v\rangle}{\|v\|^2} v,\ v\right\rangle$$
$$= \langle w, v\rangle - \frac{\langle w, v\rangle}{\|\cancel{v}\|^2}\langle \cancel{v}, v\rangle$$
$$= 0.$$

We call $\frac{\langle w, v\rangle}{\|v\|^2} v$, the orthogonal projection of w.

§ Given 2 orthogonal vectors $\mathbf{v}_1, \mathbf{v}_2$, another vector $\mathbf{w}$

How do we make up an vector $\mathbf{v}_3$ orthogonal to both $\mathbf{v}_1, \mathbf{v}_2$? In particular, the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans the same subspace as $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$?

Orthogonal projection of w is

$$\underbrace{\frac{\langle w, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w, v_2\rangle}{\|v_2\|^2} v_2}.$$

$$v_3 = w - \left(\frac{\langle w, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w, v_2\rangle}{\|v_2\|^2} v_2 \right)$$

Thus $\{v_1, v_2, v_3\}$ is orthogonal

$\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{v_1, v_2, w\}.$

§ In general

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthogonal vectors, then for any vector $\mathbf{w}$,

$$\sum_{i=1}^{n} \frac{\langle \mathbf{w}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$$

is the vector nearest to $\mathbf{w}$ in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

This is called the **orthogonal projection** of $\mathbf{w}$ onto $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Furthermore, the vector

$$\mathbf{w} - \left(\sum_{i=1}^{n} \frac{\langle \mathbf{w}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \mathbf{v}_i \right)$$

is orthogonal to each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

§ The Gram-Schmidt process

We start with any basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ for the inner product space V.

We then orthogonalize each one to the preceding ones, building up an "orthogonal basis" as we go.

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\vdots$$

$$v_n = w_n - \frac{\langle w_n, v_1 \rangle}{\|v_1\|^2} v_1 - \cdots - \frac{\langle w_n, v_{n-1} \rangle}{\|v_{n-1}\|^2} v_{n-1}$$

Thus $\{v_1, \ldots, v_n\}$ is orthogonal. #

Example. Consider the vectors $\mathbf{w}_1 = (1 \quad 1 \quad 0)^T$, $\mathbf{w}_2 = (0 \quad 1 \quad 1)^T$, and $\mathbf{w}_3 = (1 \quad 0 \quad 1)^T$, that form a basis of $\mathbb{R}^3$. To construct an orthogonal basis and an orthonormal basis using the Gram-Schmidt process.

$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$ is an orthonormal basis

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$\|v_1\|^2 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 2$$

$$\langle w_2, v_1 \rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1$$

$$v_2 = w_2 - \frac{1}{2} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}}.$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\langle w_3, v_1 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1,$$

$$\langle w_3, v_2 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\rangle = \frac{1}{2}.$$

$$\|v_2\|^2 = 3/2.$$

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

Thus, $\{v_1, v_2, v_3\}$ is orthogonal

Example. We know that 1, x, and x^2 form a basis for $\mathcal{P}^{(2)}([0,1])$, the space of polynomials of degree ≤ 2 on $[0,1]$. Let's turn them into an orthonormal basis, with respect to the usual L^2 inner product.

$$q_0 = 1$$

$$q_1(x) = x - \frac{\langle x, q_0 \rangle}{\|q_0\|^2} q_0 \quad \left(= x - \frac{1}{2} \right)$$

$$q_2(x) = x^2 - \frac{\langle x^2, q_0 \rangle}{\|q_0\|^2} q_0 - \frac{\langle x^2, q_1 \rangle}{\|q_1\|^2} q_1 \left(= x^2 - x + \frac{1}{6} \right)$$

⌜will continue on 3/25⌟