Lecture 23: Quick review from previous lecture

- The basis $\{1, x, x^2\}$ for $\mathcal{P}^{(2)}$ $([0,1])$ do NOT form an orthogonal basis.
- (Gram-Schmidt Process)
  Suppose that $\mathbf{a}_1, \cdots, \mathbf{a}_n$ are linearly independent.

Turn $\mathbf{a}_1, \cdots, \mathbf{a}_n$ to orthogonal vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

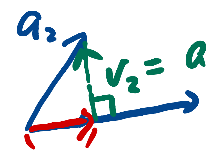
$$\mathbf{v}_1 = \mathbf{a}_1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\langle \mathbf{a}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\langle \mathbf{a}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{a}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_n = \mathbf{a}_n - \frac{\langle \mathbf{a}_n, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{a}_n, \mathbf{v}_{n-1} \rangle}{\|\mathbf{v}_{n-1}\|^2}\mathbf{v}_{n-1}$$



$v_2 = a_2 - \frac{\langle a_2, v_1 \rangle}{\|v_1\|^2} v_1$

$a_1 = v_1$

$\frac{\langle a_2, v_1 \rangle}{\|v_1\|^2} v_1$

Thus $\langle v_2, v_1 \rangle = 0$.

Today we will discuss examples for the Gram-Schmidt process and introduce the Orthogonal Matrices.

- Lecture will be recorded -

Lecture video can be found in Canvas "Media Gallery"

Example. We know that 1, x, and x^2 form a basis for $\mathcal{P}^{(2)}([0,1])$, the space of polynomials of degree ≤ 2 on $[0,1]$. Let's turn them into an orthonormal basis, with respect to the usual L^2 inner product.

(ONB)

By Gram-Schmidt process, we will turn $\{1, x, x^2\}$ into orthogonal basis.

$$\begin{cases} P_1 = 1 \\ P_2 = x - \dfrac{\langle x, P_1\rangle}{\|P_1\|^2} P_1 = \boxed{x - \frac{1}{2}} \\ P_3 = x^2 - \dfrac{\overbrace{\langle x^2, P_1\rangle}^{=1/3}}{\underbrace{\|P_1\|^2}_{=1}} P_1 - \dfrac{\overbrace{\langle x^2, P_2\rangle}^{=1/12}}{\underbrace{\|P_2\|^2}_{=1/12}} P_2 = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) \\ \qquad = \boxed{x^2 - x + \frac{1}{6}} \end{cases}$$

① $\langle x, P_1\rangle = \int_0^1 x \cdot 1 \, dx = \frac{1}{2}$

② $\|P_1\|^2 = \langle P_1, P_1\rangle = \int_0^1 1 \, dx = 1$.

③ $\langle x^2, P_1\rangle = \int_0^1 x^2 \cdot 1 \, dx = \frac{1}{3}$.

④ $\|P_2\|^2 = \int_0^1 (x - \frac{1}{2})^2 \, dx = 1/12$.

⑤ $\langle x^2, P_2\rangle = \int_0^1 x^2 (x - \frac{1}{2}) \, dx = 1/12$.

Thus, we have orthogonal basis $\{P_1, P_2, P_3\}$.

To get ONB, we do

$$\left\{ \frac{P_1}{\|P_1\|}, \frac{P_2}{\|P_2\|}, \frac{P_3}{\|P_3\|} \right\}$$

$$= \left\{ 1, \underbrace{\frac{x - \frac{1}{2}}{\frac{1}{\sqrt{12}}}}_{= \sqrt{12}(x - \frac{1}{2})}, \frac{x^2 - x + \frac{1}{6}}{\|P_3\|} \right\}$$

$\longrightarrow$ Ex. compute $\|P_3\|$.

Example. Let the subspace $W \subset \mathbb{R}^4$ consist of all vectors orthogonal to $\mathbf{a} = [1 \quad 1 \quad 1 \quad 1]^T$. Find an orthonormal basis for W.

(ONB).

1) $x = (x_1, x_2, x_3, x_4)$

pivot

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}$$

$x \cdot a = 0. \Rightarrow x_1 + x_2 + x_3 + x_4 = 0.$

free variables.

$$\Rightarrow \boxed{x_1 = -x_2 - x_3 - x_4}$$

$$W = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right\}.$$

$x_2 = 1,\ x_3 = x_4 = 0$; $x_3 = 1,\ x_2 = x_4 = 0$; $x_4 = 1,\ x_2 = x_3 = 0$

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad w_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad w_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $\{w_1, w_2, w_3\}$ is a basis for W.

($W = \text{span}\{w_1, w_2, w_3\}$).

2) To find ONB basis for W.

First, we use Gram Schmidt to find orthogonal basis

$$v_1 = w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}.$$

Next, ONB for W is

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}.$$

<u>EX</u>: To find explicit vectors. #

4.3 Orthogonal Matrices

Definition: A square matrix A is called an **orthogonal matrix** if it satisfies

$$A^TA = AA^T = I.$$

Recall: The inverse of A: $X A = A X = I$.

X (left inverse of A), X (right inverse of A).

Some Properties about an orthogonal matrix A:

- The inverse of A is

$$A^T = A^{-1}.$$

- It is easy to solve the system $A\mathbf{x} = \mathbf{v}$.

[Since $A^TA = I$, we immediately have the solution $\mathbf{x} = A^{-1}\mathbf{v} = A^T\mathbf{v}$. Thus there is no need to apply Gaussian elimination to solve the system.]

- If A is orthogonal, then $\det(A) = \pm 1$.

Γ $A^TA = I \Rightarrow \det(A^TA) = \det I$

$\Rightarrow (\det A^T)(\det A) = 1$

Moreover, we have

$\Rightarrow \det A \det A = 1 \Rightarrow \det A = \pm 1$ #

Fact 1: A square matrix A is an **orthogonal matrix** if and only if ($\Longleftrightarrow$) its columns form an orthonormal basis (ONB) on $\mathbb{R}^n$ with respect to the Euclidean dot product.

[To see this:] $A = [v_1 \ v_2 \ \cdots v_n]_{n\times n}$.

$(\Rightarrow)$ A is orthogonal.

$A^TA = I. \Rightarrow \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} [v_1 \ v_2 \ \cdots \ v_n] = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$

$\Rightarrow \begin{bmatrix} v_1^Tv_1 & v_1^Tv_2 & \cdots & v_1^Tv_n \\ v_2^Tv_1 & v_2^Tv_2 & \cdots & v_2^Tv_n \\ \cdots & \cdots & \cdots & v_n^Tv_n \end{bmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & 1 \end{pmatrix}$

Thus $v_i^Tv_j = 0$, $i \neq j$; $v_i^Tv_i = 1$, $1 \le i \le n$.

Hence $\{v_1, \ldots, v_n\}$ ONB. #

- If A is orthogonal, so is A^T (since $(A^T)^T = A$). This implies that the column vectors of A^T (they are row vectors of A) also from an orthonormal basis of $\mathbb{R}^n$. ONB.

A is orthogonal $\Rightarrow A^T$ is orthogonal $\xRightarrow{\text{Fact!}}$ columns of A^T is ONB

Example. The matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Longrightarrow \text{rows of } A^T \text{ is ONB.}$$

is orthogonal.

[To see this:] Check $A^TA = AA^T = I_2$.

$$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}.$$

since $\cos^2\theta + \sin^2\theta = 1$.

Similarly,

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

is also orthogonal. (Exercise: $B^TB = BB^T = I_2$).

[*** One can easily see that the rows (columns) of the matrix form an orthonormal basis for $\mathbb{R}^2$.] $\{(\cos\theta, \sin\theta)^T, (\sin\theta, -\cos\theta)^T\}$ is ONB for $\mathbb{R}^2$. #

Fact 2: If A and B are **orthogonal matrices**, then AB is **orthogonal** too.

[To see this:] $(AB)^T(AB) = B^T A^T A B = B^T I B = B^T B = I$.

Similarly,

$(AB)(AB)^T = I$. #.

- If A is orthogonal, then the matrix A preserve length in the sense that

$$\|A\mathbf{x}\| = \|\mathbf{x}\| \qquad \text{for all } x \in \mathbb{R}^n \qquad \text{(Homework problem)}$$