## Lecture 25: Quick review from previous lecture

- $$\underbrace{\left(\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right)}_{A} = \underbrace{\left(\begin{array}{c|c|c|c} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{array}\right)}_{Q} \underbrace{\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix}}_{R}$$

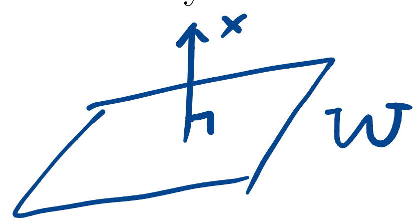
  , orthogonal

  $Q^TQ = QQ^T = I$

  where $r_{kk} = \|\mathbf{v}_k\| = \langle \mathbf{a}_k, \mathbf{q}_k \rangle$ and $r_{ij} = \langle \mathbf{a}_j, \mathbf{q}_i \rangle$. This is called the **QR factorization**.

- $\mathbf{x}$ is **orthogonal** to the subspace $W$ of $V$ if it is orthogonal to every vector in $W$, that is,

  $$\langle \mathbf{x}, \mathbf{w} \rangle = 0 \qquad \text{for all } \mathbf{w} \in W,$$

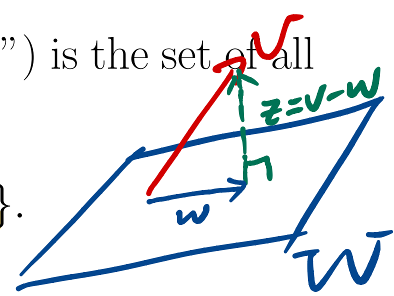
  denoted by

  $$\mathbf{x} \perp W.$$

- The **orthogonal projection** of $\mathbf{v}$ onto the subspace $W$ of $V$ is the element $\mathbf{w} \in W$ such that the difference $\mathbf{z} = \mathbf{v} - \mathbf{w}$ orthogonal to $W$.

- The **orthogonal complement** $W^\perp$ (pronounced "$W$ perp") is the set of all vectors orthogonal to $W$, that is,

  $$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

  $z \perp W$.

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Today we will discuss orthogonal projections and subspaces.

**- Lecture will be recorded -**

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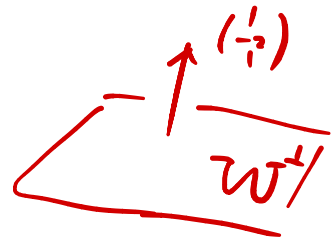
Note that the only vector contained in both $W$ and $W^\perp$ is $\mathbf{0}$.

$\ulcorner$ $x \in W$, and $x \in W^\perp$. Thus $\langle \underset{\in W}{x}, \underset{\in W^\perp}{x} \rangle = 0$

$\|x\|^2 = 0 \Rightarrow x = 0$. # $\lrcorner$

**Example.** Find $W^\perp$, the orthogonal complement to $W = \text{span}\{\mathbf{w}\}$ in $\mathbb{R}^3$, where

$$\mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$




$\vec{x} = (x, y, z) \in W^\perp$. $\quad \vec{x} \perp w$

So, $w^T \vec{x} = 0 \Rightarrow \begin{bmatrix} \underset{\text{pivot}}{1} & -2 & 1 \end{bmatrix}_{1\times 3} \vec{x} = 0$

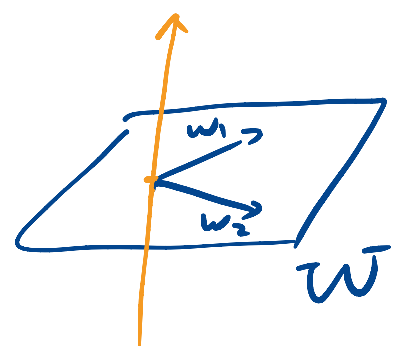
$\Rightarrow x - 2\underline{y} + \underline{z} = 0$ free variables

Thus, $W^\perp = \left\{ \begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} \middle| \ y, z \in \mathbb{R} \right\}$.

$\ulcorner \dim W^\perp = 2 \lrcorner$

**Example.** Suppose $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$




To find $W^\perp$.

$\vec{x} = (x, y, z) \in W^\perp$.

$\begin{cases} w_1^T \vec{x} = 0, \\ w_2^T \vec{x} = 0. \end{cases} \Rightarrow \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}_{2\times 3} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. pivot; $z \to$ free variable

$\begin{cases} y = z, \\ x = -3z, \end{cases} \quad W^\perp = \left\{ \begin{pmatrix} -3z \\ z \\ z \end{pmatrix} \middle| \ z \in \mathbb{R} \right\}$.

$= \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right\}$.

In general, if $W = \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a subspace of $\mathbb{R}^n$, then $W^\perp$ is the set of all vectors orthogonal to all of $\mathbf{w}_1, \ldots, \mathbf{w}_k$.

Thus, the space $W^\perp$ is precisely the kernel of the $k$-by-$n$ matrix

$$A = \begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_k^T \end{pmatrix}$$

[To see this:] Observe that

$$\mathbf{0} = \begin{pmatrix} \mathbf{w}_1^T\mathbf{x} \\ \vdots \\ \mathbf{w}_k^T\mathbf{x} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_k^T \end{pmatrix}}_{A} \mathbf{x}$$

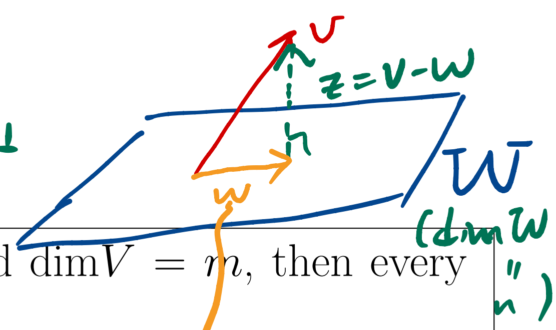
Thus, $\mathbf{x}$ is in the kernel of $A$ if and only if $\mathbf{w}_j^T\mathbf{x} = 0$ for all $j$, i.e. $\mathbf{x}$ is orthogonal to $W = \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$.

$x \in W^\perp$

$V$ ($\dim V = m$)

$z \perp W$

$\Rightarrow z \in W^\perp$




**Fact:** If $W$ is a subspace of $V$ with $\dim W = n$ and $\dim V = m$, then every vector $\mathbf{v} \in V$ can be **uniquely** decomposed into

$$\boxed{\mathbf{v} = \mathbf{w} + \mathbf{z}}$$

$W$ $\quad W^\perp$.

where $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$.

orthogonal projection of $v$ onto $W$.

Moreover, we have

$$\dim W^\perp = m - n$$

and thus,

$$\dim V = \dim W + \dim W^\perp.$$

**Example.** Let $W = \operatorname{img} A$, where

$$A = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find $W^\perp$, that is, $(\operatorname{img} A)^\perp$.

1. Find basis for $W = \operatorname{img} A$.

$$A = \begin{pmatrix} \underline{v_1} & \underline{v_2} & v_3 \end{pmatrix} \xrightarrow[\text{(G.E.)}]{\text{Gaussian Elimination}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

pivots

Basis for $\operatorname{img} A = \{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$.

2. Find $W^\perp$.

$$\begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$z \to$ free variable

$$W^\perp = \left\{ \begin{pmatrix} -z \\ z \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

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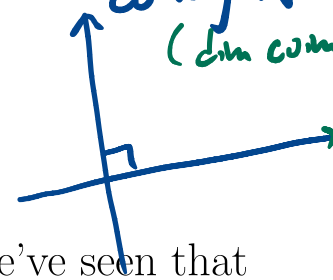
**Fact:** If $W$ is a subspace of $V$ with $\dim W = n < \infty$, then

$$(W^{\perp})^{\perp} = W.$$




Recall that:

Suppose $A = A_{m\times n}$ is any matrix with $rank A = r$. We've seen that

$$\dim \operatorname{coimg} A = r \quad \text{and} \quad \dim \ker A = n - r.$$

**Fact:** Let $A$ be any real $m \times n$ matrix. Then

$$\operatorname{coimg} A = (\ker A)^{\perp} \quad (\text{and } \ker A = (\operatorname{coimg} A)^{\perp}).$$

[To see this:] Recall $\operatorname{coimg} A = \operatorname{img} A^T$, (span of rows of $A$).

$$A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix}$$

If $x \in \ker A$, then $Ax = 0$.

$$\begin{bmatrix} r_1^T \\ \vdots \\ r_m^T \end{bmatrix} x = 0$$

So, $\ker A$ is the subspace of all vectors that are "orthogonal to <u>rows</u> of $A$".

that is, $\ker A \perp$ (span of rows) $\Rightarrow \ker A \perp \operatorname{coimg} A$

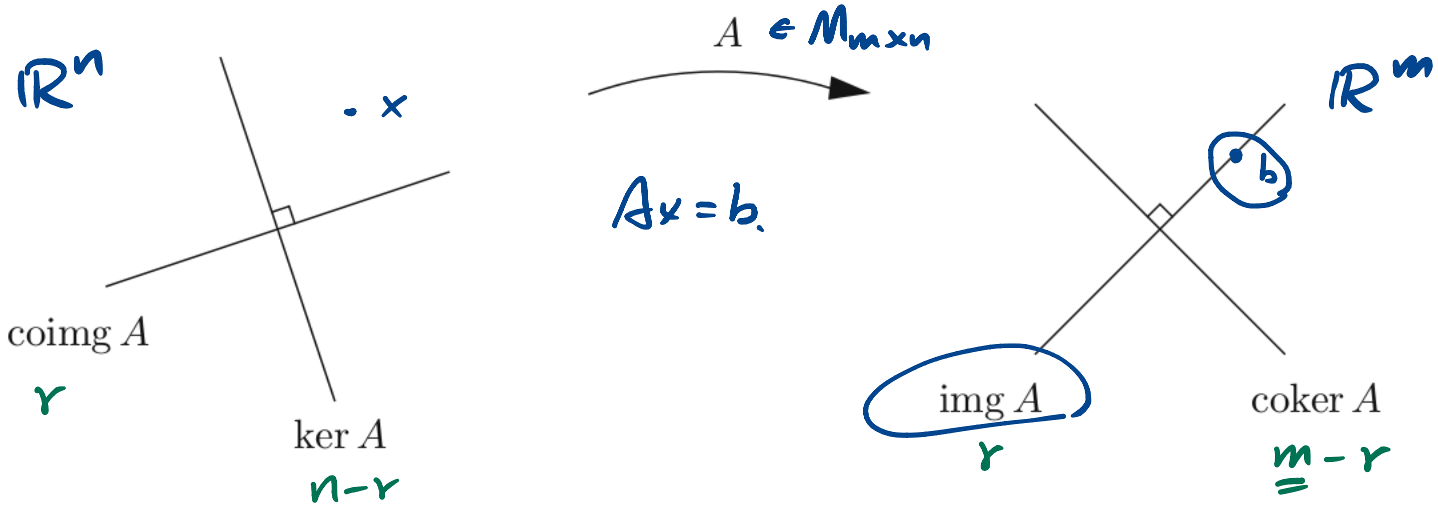
$\Rightarrow (\ker A)^{\perp} = \operatorname{coimg} A$ #

Similarly, applying the same reasoning to $A^T$, we find that

**Fact:** Let $A$ be any real $m \times n$ matrix. Then

$$\operatorname{img} A = (\operatorname{coker} A)^{\perp} \quad (\text{and } \operatorname{coker} A = (\operatorname{img} A)^{\perp}).$$

**Fact:** **[Fredholm alternative]**
The linear system $A\mathbf{x} = \mathbf{b}$ has a solution (it is compatible) $\Leftrightarrow$ $b \perp \operatorname{coker} A$




**Example.** Find the compatibility condition on the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}.$$

$(b \perp \underline{\underline{\operatorname{coker} A}})$.

1. Find a basis for coker $A$. ($ 2.5)

$= \boxed{\ker A^T}$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}. \qquad A^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$A^T \xrightarrow{G.E.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow y = -z, \quad x = -z.$$

$$\operatorname{coker} A = \left\{ \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

By Fact above,

[Example continue]

$(b_1, b_2, b_3)^T$

2. $b \perp \operatorname{coker} A$. $\langle b, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \rangle = 0$.

$$\boxed{-b_1 - b_2 + b_3 = 0}.$$

compatibility condition. #

✓ **Remark to the previous example.**

[The same compatibility condition can also be obtained by using Gaussian Elimination to solve the augment system $(A|\mathbf{b})$.]

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 3 & 2 & b_3 \end{array}\right).$$

$$\xrightarrow{③-①} \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_3 - b_1 \end{array}\right)$$

$$\xrightarrow{③-②} \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right).$$

Thus, $Ax = b$ has solutions if

$$\boxed{b_3 - b_1 - b_2 = 0}$$

compatibility condition