Lecture 25: Quick review from previous lecture

- Lecture will be recorded -

Today we will discuss orthogonal projections and subspaces.

where \( r_{kk} = \|v_k\| = \langle a_k, q_k \rangle \) and \( r_{ij} = \langle a_j, q_i \rangle \). This is called the QR factorization.

- **x** is **orthogonal** to the subspace \( W \) of \( V \) if it is orthogonal to every vector in \( W \), that is,

\[
\langle x, w \rangle = 0 \quad \text{for all } w \in W,
\]

denoted by

\[ x \perp W. \]

- The **orthogonal projection** of \( v \) onto the subspace \( W \) of \( V \) is the element \( w \in W \) such that the difference \( z = v - w \) orthogonal to \( W \).

- The **orthogonal complement** \( W^\perp \) (pronounced “\( W \) perp”) is the set of all vectors orthogonal to \( W \), that is,

\[
W^\perp = \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}.
\]
Note that the only vector contained in both $W$ and $W^\perp$ is $0$.

Thus $\langle x, x \rangle = 0$ for $x \in W$ and $x \in W^\perp$.

Example. Find $W^\perp$, the orthogonal complement to $W = \text{span}\{w\}$ in $\mathbb{R}^3$, where

$$w = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\vec{x} = (x, y, z) \in W^\perp \iff \vec{x} \perp w$

So, $w^\top \vec{x} = 0 \iff \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x - 2y + z = 0$

Thus, $W^\perp = \left\{ \begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$.

Example. Suppose $W = \text{span}\{w_1, w_2\}$, where

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

To find $W^\perp$,

$$\vec{x} = (x, y, z) \in W^\perp \iff \begin{pmatrix} w_1^\top \\ w_2^\top \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x - 2y + z = 0$$

Thus, $W^\perp = \left\{ \begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$. 

$\dim W^\perp = 2$.
In general, if $W = \text{span}\{w_1, \ldots, w_k\}$ is a subspace of $\mathbb{R}^n$, then $W^\perp$ is the set of all vectors orthogonal to all of $w_1, \ldots, w_k$.

Thus, the space $W^\perp$ is precisely the kernel of the $k$-by-$n$ matrix

$$A = \begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_k^T \end{pmatrix}$$

[To see this:] Observe that

$$\mathbf{0} = \begin{pmatrix} \mathbf{w}_1^T \mathbf{x} \\ \vdots \\ \mathbf{w}_k^T \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_k^T \end{pmatrix} \mathbf{x}$$

Thus, $\mathbf{x}$ is in the kernel of $A$ if and only if $\mathbf{w}_j^T \mathbf{x} = 0$ for all $j$, i.e. $\mathbf{x}$ is orthogonal to $W = \text{span}\{w_1, \ldots, w_k\}$.

Fact: If $W$ is a subspace of $V$ with $\dim W = n$ and $\dim V = m$, then every vector $\mathbf{v} \in V$ can be uniquely decomposed into

$$\mathbf{v} = \mathbf{w} + \mathbf{z}$$

where $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$.

Moreover, we have

$$\dim W^\perp = m - n$$

and thus,

$$\dim V = \dim W + \dim W^\perp.$$
Example. Let $W = \text{img } A$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

Find $W^\perp$, that is, $(\text{img } A)^\perp$.

1. Find basis for $W = \text{img } A$.

\[ A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Basis for $\text{img } A = \{ \mathbf{v}_1, \mathbf{v}_2 \} = \{ (1), (1) \}$.

$W = \text{span} \{ (1), (1) \}$.

2. Find $W^\perp$.

\[ \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

\[ \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

$W^\perp = \{ (\mathbf{0}) \mid \mathbf{z} \in \mathbb{R} \} = \text{span} \{ (1) \}$.
**Fact:** If $W$ is a subspace of $V$ with $\dim W = n < \infty$, then

$$(W^\perp)^\perp = W.$$ 

Recall that:
Suppose $A = A_{m \times n}$ is any matrix with $\text{rank} A = r$. We’ve seen that

$$\dim \text{coimg } A = r \quad \text{and} \quad \dim \text{ker } A = n - r.$$ 

**Fact:** Let $A$ be any real $m \times n$ matrix. Then

$$\text{coimg } A = (\ker A)^\perp \quad \text{(and ker } A = (\text{coimg } A)^\perp).$$

[To see this:] Recall $\text{coimg } A = \text{img } A^T, (\text{span of rows of } A)$.

$$A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix}$$

If $x \in \ker A$, then $Ax = 0$.

$$\begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} x = 0$$

So, $\ker A$ is the subspace of all vectors that are “orthogonal to rows of $A$”.

That is, $\ker A \perp (\text{span of rows}) \Rightarrow \ker A \perp \text{coimg } A \Rightarrow (\ker A)^\perp = \text{coimg } A$.

Similarly, applying the same reasoning to $A^T$, we find that

**Fact:** Let $A$ be any real $m \times n$ matrix. Then

$$\text{img } A = (\text{coker } A)^\perp \quad \text{(and coker } A = (\text{img } A)^\perp).$$
Fact: [Fredholm alternative]
The linear system $A\mathbf{x} = \mathbf{b}$ has a solution (it is compatible) $\iff \mathbf{b} \perp \text{coker } A$

Example. Find the compatibility condition on the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}.$$

1. Find a basis for $\text{coker } A$. ($\text{rref } A^T$)

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \quad A^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$A^T \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y = -2, \quad x = -2.$

$\text{coker } A = \{ (\begin{pmatrix} -2 \\ z \\ z \end{pmatrix} | z \in \mathbb{R} ) \} = \text{span}\{ (\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} ) \}.$
Remark to the previous example.

[The same compatibility condition can also be obtained by using Gaussian Elimination to solve the augment system \((A|b)\).

\[
(A \mid b) = \begin{pmatrix}
1 & 2 & 1 & b_1 \\
0 & 1 & 1 & b_2 \\
1 & 3 & 2 & b_3
\end{pmatrix}.
\]

\[
\begin{align*}
\text{3-2} & \rightarrow \\
\begin{pmatrix}
1 & 2 & 1 & b_1 \\
0 & 1 & 1 & b_2 \\
0 & 0 & 1 & b_3 - b_1 - b_2
\end{pmatrix}.
\end{align*}
\]

Thus, \(Ax=b\) has solutions if

\[
b_3 - b_1 - b_2 = 0.
\]