Lecture 26: Quick review from previous lecture

- If $W$ is a subspace of $V$ with $\dim W = n$ and $\dim V = m$, then every vector $v \in V$ can be uniquely decomposed into $v = w + z$ where $w \in W$ and $z \in W^\perp$. Moreover, $\dim V = \dim W + \dim W^\perp$.

- $\text{coimg } A = (\ker A)^\perp$ and $\img A = (\coker A)^\perp$.
- The linear system $Ax = b$ has a solution (it is compatible) $\iff b \perp \coker A$.
- The only vector in both $W$ and $W^\perp$ is zero element $0$.

Today we will discuss linear functions.

- Lecture will be recorded -

Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas.
Fact:
(1) Let $A$ be any real $m \times n$ matrix. Any vector $b$ in $\text{img } A$ has “exactly one” vector from $\text{coimg } A$ mapping from it.

(2) Moreover, if $\{v_1, \cdots, v_r\}$ is a basis of $\text{coimg } A$, then
\[
\{Av_1, \cdots, Av_r\} \text{ is a basis of } \text{img } A.
\]

[To see this:]

(1) $b \in \text{img } A$. So we can find $v \in \mathbb{R}^n$ so that $Av = b$.

\[\text{(coimg } A)^\perp = \text{ker } A.\]

Thus we write
\[V = x + z \in \text{coimg } A \subseteq (\text{ker } A) = (\text{coimg } A)^\perp.
\]

\[\text{Av} = A(x + z) = Ax + A\overline{z} = Ax.
\]

Thus $b = Ax$, where $x \in \text{coimg } A$.

(Unique): Suppose $Ax_1 = Ax_2 = b$, $x_1, x_2 \in \text{coimg } A$.

\[A(x_1 - x_2) = 0.\]

Thus, we also have

Fact: A compatible linear system $Ax = b$ with $b \in \text{img } A$ has a unique solution $x^* \in \text{coimg } A$ satisfying $Ax^* = b$.

The general solution is $x = x^* + z$, where $x^* \in \text{coimg } A$ and $z \in \text{ker } A$. Then $x^*$ has the smallest norm of all the solutions to $Ax = b$.

Any solution $x = x^* + z$.

\[
\begin{align*}
11x_1^2 &= 11x_1^2 + z_1^2 = 11x^*_1 + 2\langle x^*, z \rangle \\
&\leq 11x^*_1 + 11z_1^2 \\
&\leq 11x^*_1 + 11z_1^2
\end{align*}
\]

Thus, $11x_1^2 \geq 11x^*_1$.
To find the solution of minimum Euclidean norm, that is, \( x^* \):

1. Using Gaussian Elimination to find the general solution \( x \) to the system \( Ax = b \).

2. Finding the basis \( v_1, \ldots, v_\ell \) for \( \text{ker} \ A \), and then using the conditions \( v_j^T x = 0 \).

**Example.** Find the solution of minimum Euclidean norm \( x^* \) of the linear system \( Ax = b \), where

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}.
\]

and \( b = (1, 1, 2)^T \).

1. Find general solutions for \( Ax = b \). 

   **augmented system:**

   \[
   (A \ | \ b) = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 2 \end{pmatrix}.
   \]

   G.E. \[
   \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
   \]

   \( y = 1 - z \)

   \( x = 1 - 2y - 3z = -1 - z \).

   **General solution:** \( \{( \begin{pmatrix} -1-z \\ 1-z \\ z \end{pmatrix} \ | \ z \in \mathbb{R} \} \)

2. Find basis for \( \text{ker} \ A \).

   \[
   \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
   \]

   So, \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} -z \\ z \\ z \end{pmatrix} \) if \( z = 1 \).
\[ \ker A = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}. \]

3. \[ \left\langle \begin{pmatrix} -1 & -2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle = 0. \]

Thus, \[ z = 0. \]

It implies that \[ x^* = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \]

has the smallest norm. \#
Chapter 7 Linearity

7.1 Linear Functions

**Definition: [Linear operators]**
If $L : V \rightarrow W$ is a mapping between vector spaces $V$ and $W$, we say that $L$ is **linear** if for all vectors $x$ and $y$ in $V$, and scalars $c$ such that

1. $L[cx] = cL[x]$

We call such a mapping $L$ a **linear operator**. We call $V$ the **domain** for $L$, and $W$ the **codomain**.

We may also say $L$ is a **linear function**, or a **linear map** (or mapping), or a **linear transformation**. They all refer to the same properties.

**Properties:**

- For any scalars $c$ and $d$ and any vectors $x$ and $y$ in $V$,
  $$L[cx + dy] = cL[x] + dL[y]$$

- For any scalars $c_1, \ldots, c_n$ and any vectors $x_1, \ldots, x_n$ in $V$, then
  $$L[c_1x_1 + \cdots + c_nx_n] = c_1L[x_1] + \cdots + c_nL[x_n].$$

- $L[0] = 0$ (the $0$ on the left is the zero element in $V$; the $0$ on the right is the zero element in $W$).

  To see this:
  $$L[x + y] = L[x] + L[y].$$
  \[ x = 0, \quad y = 0. \]
  $$L[0+0] = L[0] + L[0].$$
  Thus, $0 = L[0]$
Example.

1. Let \( C^0([a, b]) \) be the vector space of continuous functions on the interval \([a, b]\). Define the operator \( L \) by

\[
L[f](x) = \int_a^x f(t) dt.
\]

In other words, we have defined \( L[f] \) to be a function, that is, the integral of \( f \).

Check \( L \) is Linear:

\[
\begin{align*}
\text{① } L[cf](x) &= \int_a^x cf(t) dt = c \int_a^x f(t) dt = cL[f](x), \\
\text{② } L[f+g](x) &= \int_a^x (f+g)(t) dt = \int_a^x f(t) dt + \int_a^x g(t) dt = L[f](x) + L[g](x).
\end{align*}
\]

2. Now define the operator \( L : C^1([a, b]) \rightarrow C^0([a, b]) \) by

\[
L[f](x) = \frac{d}{dx} f(x) = f'(x),
\]

where \( f \) is in \( C^1([a, b]) \), the space of differentiable functions on \([a, b]\).

Check \( L \) is Linear:

\[
\begin{align*}
\text{① } L(cf)(x) &= (cf)'(x) = cf'(x) = cL[f](x), \\
\text{② } L[f+g](x) &= (f+g)'(x) = f'(x) + g'(x) = L[f] + L[g].
\end{align*}
\]
Q: What are the linear operators $L : \mathbb{R} \to \mathbb{R}$?

Suppose $L$ is any linear operator $L : \mathbb{R} \to \mathbb{R}$. Then $L[cx] = cL[x]$ for any numbers $c$ and $x$ (we think of $c$ as a scalar and $x$ as a vector, but since $x$ is in $\mathbb{R}$ they’re both just numbers).

For any $x \in \mathbb{R}$,

$$L[x] = L[\frac{x}{1}] = xL[1].$$

Let $\alpha = L[1]$, scalar. So the linear operator $L[x] = \alpha x$, for fixed scalar $\alpha$. 

Remark: All linear operators $L : \mathbb{R} \to \mathbb{R}$ are lines passing through the origins. ($L[x] = ax$)

Warning: The function $f(x) = ax + b$ is not a linear function unless $b = 0$, even though its graph is also a line; this is because $f(0) = b$, so it doesn’t pass through the origin (unless $b = 0$)

Example. We can think of $A$ ($m \times n$ matrix) as defining a mapping $L$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by $L[v] = Av$ $v \in \mathbb{R}^n$.

The mapping $L$ is linear (that is, a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$).


$\blacksquare$
Q: Are there any other linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$? That is, can there be a linear mapping not of the above form, for some matrix $A$?

Fact 1: Every linear mapping $L$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication, $L[v] = Av$, where $A$ is an $m \times n$ matrix.

[To see this:]

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$.

For $j = 1, \ldots, n$,

$$L[e_j] = \hat{a}_j = \left( \begin{array}{c} a_{1j} \\ \vdots \\ a_{mj} \end{array} \right) = a_{1j} \hat{e}_1 + \cdots + a_{mj} \hat{e}_m.$$ 

Construct a $(m \times n)$ matrix

$$A = \left( \begin{array}{ccc} \hat{a}_1 & \cdots & \hat{a}_n \end{array} \right) = \left( \begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{array} \right)_{m \times n}.$$ 

Any $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, 

$$v = v_1 e_1 + \cdots + v_n e_n$$ 

$$L[v] = v_1 L[e_1] + \cdots + v_n L[e_n]$$ 

$$= v_1 \hat{a}_1 + \cdots + v_n \hat{a}_n$$ 

$$= A \left( \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} \right) = A v.$$