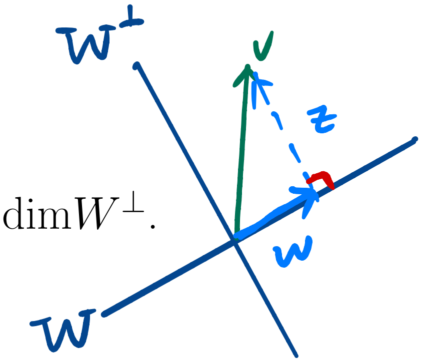
Lecture 26: Quick review from previous lecture

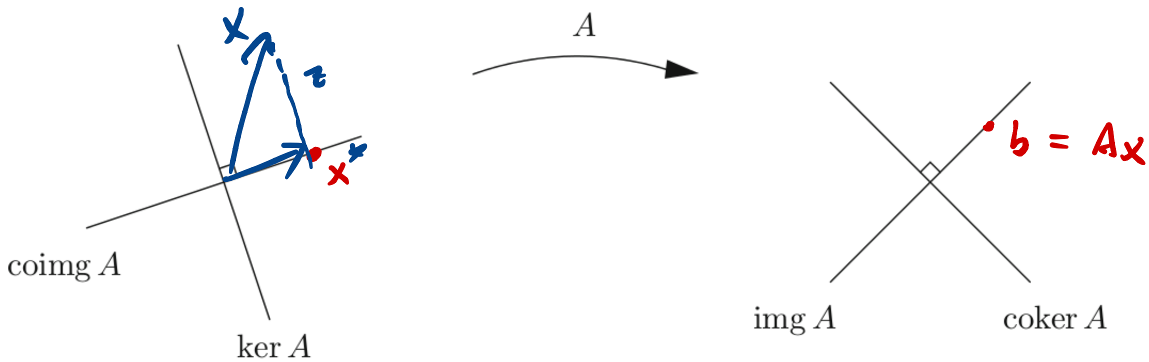
- If W is a subspace of V with $\dim W = n$ and $\dim V = m$, then every vector $\mathbf{v} \in V$ can be **uniquely** decomposed into

$$\mathbf{v} = \mathbf{w} + \mathbf{z}$$

  where $\mathbf{w} \in W$ and $\mathbf{z} \in W^{\perp}$. Moreover, $\dim V = \dim W + \dim W^{\perp}$.



- coimg $A = (\ker A)^{\perp}$ and img $A = (\text{coker } A)^{\perp}$.
- The linear system $A\mathbf{x} = \mathbf{b}$ has a solution (it is compatible) $\Leftrightarrow b \perp \text{coker } A$.
- The only vector in both W and $W^{\perp}$ is zero element $\mathbf{0}$.



Today we will discuss linear functions.

- Lecture will be recorded -

Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas.

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m.$$

Fact:

(1) Let A be any real $m \times n$ matrix. Any vector b in img A has "exactly one" vector from coimg A mapping from it.

(2) Moreover, if $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is a basis of coimg A, then

(HW)

$$\{A\mathbf{v}_1, \cdots, A\mathbf{v}_r\} \text{ is a basis of img } A.$$

[To see this:]

(1) $b \in \operatorname{img} A$. So we can find $\underline{\underline{v}} \in \mathbb{R}^n$ so that $Av = b$.

$(\operatorname{coimg} A)^\perp = \ker A$. Thus we write

$$\underset{\operatorname{coimg} A}{\underset{\cap}{}}\ v = \underset{\substack{\cap \\ \operatorname{coimg} A}}{x} + \underset{\in (\ker A) = (\operatorname{coimg} A)^\perp}{z}.$$

$$\begin{cases} Av = A(x+z) = Ax + \cancel{A\underset{\ker A}{z}}^{\,0} = Ax. \\ Av = b. \end{cases}$$

Thus $b = Ax$, where $x \in \operatorname{coimg} A$.

(unique): Suppose $Ax_1 = Ax_2 = b$, $x_1, x_2 \in \operatorname{coimg} A$

$A(x_1 - x_2) = 0$. $(x_1 - x_2) \in \ker A$ also $x_1 - x_2 \in \operatorname{coimg} A$

Thus, we also have

Fact: A compatible linear system $A\mathbf{x} = \mathbf{b}$ with $b \in \operatorname{img} A$ has a unique solution $\mathbf{x}^* \in \operatorname{coimg} A$ satisfying $A\mathbf{x}^* = \mathbf{b}$.

The general solution is $\mathbf{x} = \mathbf{x}^* + \mathbf{z}$, where $\mathbf{x}^* \in \operatorname{coimg} A$ and $\mathbf{z} \in \ker A$. Then $\mathbf{x}^*$ has the smallest norm of all the solutions to $A\mathbf{x} = \mathbf{b}$.

Any solution $x = x^* + z$.

$$\|x\|^2 = \|x^* + z\|^2 = \|x^*\|^2 + 2\langle \underset{\operatorname{coimg} A}{x^*}, \underset{\ker A}{z}\rangle^{0} + \|z\|^2$$

$$= \|x^*\|^2 + \|z\|^2$$

$$\geq \|x^*\|^2$$

Thus, $\|x\| \geq \|x^*\|$. #

$x_1 - x_2 \in \ker A$

$\{ x_1 - x_2 \in (\ker A)^\perp = \operatorname{coimg} A$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ #

To find the solution of minimum Euclidean norm, that is, $\mathbf{x}^*$:

1. Using Gaussian Elimination to find the general solution $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$.

2. Finding the basis $v_1, \cdots, v_\ell$ for $\ker A$, and then using the conditions $v_j^T \mathbf{x} = 0$.

Example. Find the solution of minimum Euclidean norm $\mathbf{x}^*$ of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}.$$

and $\mathbf{b} = (1, 1, 2)^T$.

1. Find general solutions for $Ax = b$.

augmented system:

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 2 \end{array}\right).$$

$$\xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$y = 1 - z$

$x = 1 - 2y - 3z = -1 - z.$

General solution: $\left\{ \begin{pmatrix} -1-z \\ 1-z \\ z \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\}$

2. Find basis for ker A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix}$, $z = 1$.

[Example Continue]

$$\ker A = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

3. $$\left\langle \begin{pmatrix} -1-z \\ 1-z \\ z \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0.$$

Thus, $\underline{z = 0}$. $\begin{pmatrix} -1-z \\ 1-z \\ z \end{pmatrix}$

It implies that $x^* = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

has the smallest norm. #

Chapter 7 Linearity

7.1 Linear Functions

V domain $\xrightarrow{L}$ W codomain

Definition: [Linear operators]
If $L : V \to W$ is a mapping between vector spaces V and W, we say that L is **linear** if for all vectors $\mathbf{x}$ and $\mathbf{y}$ in V, and scalars c such that

$$\text{①} \quad L[c\mathbf{x}] = cL[\mathbf{x}]$$
$$\text{②} \quad L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}].$$

We call such a mapping L a **linear operator**. We call V the **domain** for L, and W the **codomain**.

We may also say L is a *linear function*, or a *linear map* (or mapping), or a *linear transformation*. They all refer to the same properties.

Properties:

- For any scalars c and d and any vectors $\mathbf{x}$ and $\mathbf{y}$ in V,

$$L[\underline{c\mathbf{x} + d\mathbf{y}}] = \underline{c}L[\mathbf{x}] + \underline{d}L[\mathbf{y}]$$

- For any scalars $c_1, \cdots, c_n$ and any vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ in V, then

$$L[c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n] = \underline{c_1}L[\mathbf{x}_1] + \cdots + \underline{c_n}L[\mathbf{x}_n].$$

- $L[\mathbf{0}] = \mathbf{0}$ (the $\mathbf{0}$ on the left is the zero element in V; the $\mathbf{0}$ on the right is the zero element in W).

[To see this:] $L[x+y] = L[x] + L[y]$.

$x = 0,\ y = 0$.

$$\underset{\underset{\cancel{L[0]}}{\parallel}}{L[0+0]} = \cancel{L[0]} + L[0].$$

Thus, $\underset{\text{zero element in } W}{0} = \underset{\text{zero element in } V}{L[0]}$

Example.

1. Let $C^0([a,b])$ be the vector space of continuous functions on the interval $[a,b]$. Define the operator L by

$$L[f](x) = \int_a^x f(t)dt.$$

In other words, we have defined $L[f]$ to be a function, that is, the integral of f.

Check L is Linear:

① $L[cf](x) = \int_a^x \underset{\text{constant}}{c} f(t)\,dt = c\int_a^x f\,dt$
$= cL[f](x)$.

② $L[f+g](x) = \int_a^x f(t)+g(t)\,dt = \int_a^x f\,dt + \int_a^x g\,dt$
$= L[f](x) + L[g](x)$.

2. Now define the operator $L : C^1([a,b]) \to C^0([a,b])$ by

$$L[f](x) = \frac{d}{dx}f(x) = f'(x),$$

where f is in $C^1([a,b])$, the space of differentiable functions on $[a,b]$.

Check L is Linear!

① $L[cf](x) = (cf)'(x) = cf'(x) = cL[f](x)$.

constant

② $L[f+g](x) = (f+g)'(x) = f'(x) + g'(x)$
$= L[f] + L[g]$. #

Q: What are the linear operators $L : \mathbb{R} \to \mathbb{R}$?

Suppose L is any linear operator $L : \mathbb{R} \to \mathbb{R}$. Then $L[cx] = cL[x]$ for any numbers c and x (we think of c as a scalar and x as a vector, but since x is in $\mathbb{R}$ they're both just numbers).

For any $x \in \mathbb{R}$,

$$L[x] = L[\underbrace{x \cdot 1}_{\text{number}}] = x\, L[1].$$

Let $a = L[1]$, scalar. So the linear operator

$$L[x] = \underbrace{a}_{(1\times 1 \text{ matrix})} x, \text{ for fixed scalar } a. \quad \#$$

Remark: All linear operators $L : \mathbb{R} \to \mathbb{R}$ are lines passing through the origins. ($L[x] = ax$) #

Warning: The function $f(x) = ax + b$ is **not** a linear function unless $b = 0$, even though its graph is also a line; this is because $f(0) = b$, so it doesn't pass through the origin (unless $b = 0$)

Example. We can think of A ($m \times n$ matrix) as defining a mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined by

$$\boxed{L[\mathbf{v}] = A\mathbf{v} \quad v \in \mathbb{R}^n.}$$

The mapping L is linear (that is, a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$).

① $L[cv] = cL[v]$.

② $L[v+w] = L[v] + L[w]$. #

Q: Are there any other linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$? That is, can there be a linear mapping not of the above form, for some matrix A?

> **Fact 1:** Every linear mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication, $L[\mathbf{v}] = A\mathbf{v}$, where A is an $m \times n$ matrix.

[To see this:]

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$

$\hat{e}_1, \ldots, \hat{e}_m$ " for $\mathbb{R}^m$.

For $j = 1, \ldots, n$,

$$L[e_j] = \vec{a}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = a_{1j}\hat{e}_1 + \cdots + a_{mj}\hat{e}_m.$$

Construct a $(m \times n)$ matrix

$$A = \left(\vec{a}_1 \cdots \vec{a}_n\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}.$$

Any $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$,

$$= v_1 e_1 + \cdots + v_n e_n$$

$$L[v] = v_1 L[e_1] + \cdots + v_n L[e_n]$$

$$= v_1 \vec{a}_1 + \cdots + v_n \vec{a}_n$$

$$= A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = Av. \quad \#$$