Lecture 27: Quick review from previous lecture

- A compatible linear system $A\mathbf{x} = \mathbf{b}$ with $b \in \operatorname{img} A$ has a unique solution $\mathbf{x}^* \in \operatorname{coimg} A$ satisfying $A\mathbf{x}^* = \mathbf{b}$.

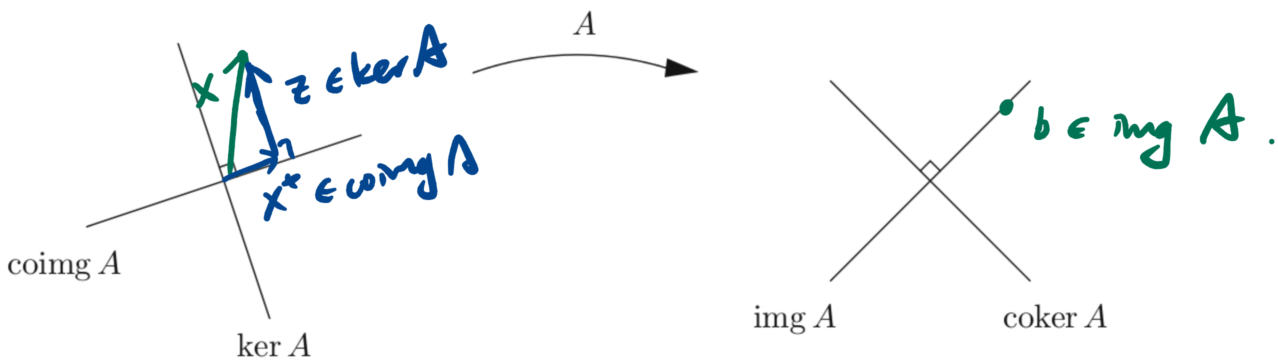
  The general solution is $\mathbf{x} = \mathbf{x}^* + \mathbf{z}$, where $\mathbf{x}^* \in \operatorname{coimg} A$ and $\mathbf{z} \in \ker A$. Then $\mathbf{x}^*$ has the smallest norm of all the solutions to $A\mathbf{x} = \mathbf{b}$.

- We call $L : V \to W$ is a linear mapping if for all vectors $\mathbf{x}$ and $\mathbf{y}$ in V, and scalars c such that

$$L[c\mathbf{x}] = cL[\mathbf{x}], \qquad L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}].$$

- Every linear mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication, $L[\mathbf{v}] = A\mathbf{v}$, where A is an $m \times n$ matrix.

$L: \mathbb{R} \to \mathbb{R}$, $L[x] = ax$, a is fixed constant.



Today we will discuss linear functions.

- Lecture will be recorded -

- Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas. See category "Announcements".
- Solutions for HW 5 and HW 6 are posted on Canvas. Only the solutions of these two HWs will be provided since they cannot be returned in this semester due to the closure of the campus.

§ The space of Linear Functions $\mathcal{L}(V, W)$.

Let $\mathcal{L}(V, W)$ be the set of all linear functions L mapping from vector space V to vector space W.

- Add two linear operators $L_1, L_2 \in \mathcal{L}(V, W)$ together:

$$(L_1 + L_2)[\mathbf{x}] = L_1[\mathbf{x}] + L_2[\mathbf{x}].$$

  Then $L_1 + L_2$ is a linear operator.

- If $L \in \mathcal{L}(V, W)$ is a linear operator and a is a scalar, we can define the new linear operator

$$(aL)[\mathbf{x}] = aL[\mathbf{x}]$$

- the zero element of $\mathcal{L}(V, W)$ is the zero function $O[\mathbf{v}] = \mathbf{0}$. ↳ zero element in W.

Thus, "$\mathcal{L}(V, W)$ is a vector space", see Definition 2.1 in textbook for the definition of a vector space.

$L: \mathbb{R}^n \to \mathbb{R}^m$, $L[x] = Ax$.

Combining with Fact 1, we have

> **Fact 2:** If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then the space $\mathcal{M}_{m\times n}$ of all $m \times n$ matrices is a vector space. (which is a fact we already knew.)

Example. The space of all linear transformations of the plane, $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, is indeed $\mathcal{M}_{2\times 2}$. And its standard basis are

$\dim M_{2\times2} = 4$.

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For any $A \in M_{2\times2}$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{a}\, E_1 + \underline{b}\, E_2 + \underline{c}\, E_3 + \underline{d}\, E_4.$$

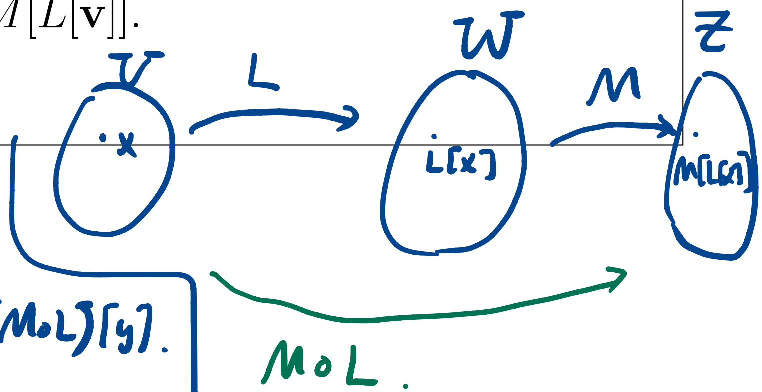
EX: $\tilde{M} = \{2\times2$ symmetry matrices$\}$. $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

§ Composition.

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$\dim \tilde{M} = 3$. #

> **Fact 3:** If $L : V \to W$ is a linear operator and $M : W \to Z$ is another linear operator, then we can define their **composition** $M \circ L : V \to Z$ by
>
> $$(M \circ L)[\mathbf{v}] = M[L[\mathbf{v}]].$$
>
> Then $(M \circ L)$ is linear.



For c, d scalars, x, y in V,

To show

$(M \circ L)[cx + dy] = c(M \circ L)[x] + d(M \circ L)[y]$.

$$(M \circ L)[cx + dy]$$

$$= M[L[cx+dy]] = M[c\overbrace{L[x]}^{\text{in } W} + d\overbrace{L[y]}^{\text{in } W}]$$

$$= c\,M[L[x]] + d\,M[L[y]].$$

$$= c\,(M \circ L)[x] + d\,(M \circ L)[y].$$ #

Example. If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $Z = \mathbb{R}^k$, then $L[\mathbf{v}] = A\mathbf{v}$ and $M[\mathbf{w}] = B\mathbf{w}$ for some matrices $A = A_{m\times n}$ and $B = B_{k\times m}$.

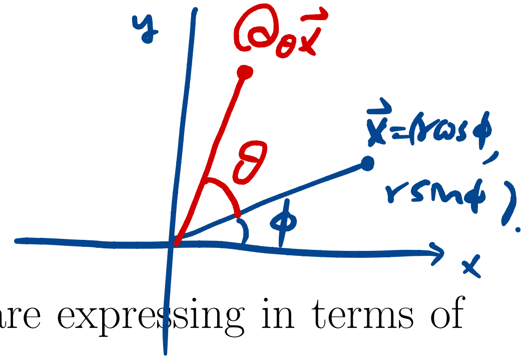
Consequently, the composition is given by

$$(M \circ L)[\mathbf{v}] = M[L[\mathbf{v}]] = M[A\mathbf{v}] = (BA)\mathbf{v}$$

In other words, multiplying two matrices corresponds to composition of the corresponding linear transformations.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Lv=Av"] \arrow[rr, bend right, "(M\circ L)[v]=(BA)v"'] & \mathbb{R}^m \arrow[r, "Mv=Bv"] & \mathbb{R}^k \end{tikzcd}$$

Example. Previously, we saw 2D rotation matrices



$$Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

If $\mathbf{x} = (r\cos\phi, r\sin\phi)$ is some vector in $\mathbb{R}^2$ (which we are expressing in terms of its polar coordinates), then find $Q_\theta \mathbf{x}$.

$$Q_\theta x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix}.$$

Thus, applying Q_θ to a vector in $\mathbb{R}^2$ is equivalent to rotate the vector counterclockwise by angle θ.

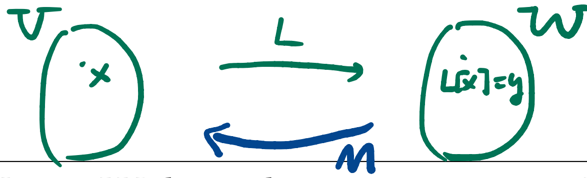
If we have two rotation matrices Q_θ and Q_ψ, then their product is

$$Q_\theta Q_\psi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

$$= \begin{pmatrix} \cos(\underline{\theta+\psi}) & -\sin(\underline{\theta+\psi}) \\ \sin(\underline{\theta+\psi}) & \cos(\underline{\theta+\psi}) \end{pmatrix}$$

$$= Q_{\theta+\psi}$$

§ Inverses



$M[y] = x.$

$L[x] = y.$

> **Definition:** Let $L : V \to W$ be a linear operator. If $M : W \to V$ is an operator such that
>
> $(M \circ L)[x] = I_V[x] = x.$ $\quad$ $= M[y]$
>
> $$M \circ L = I_V, \quad L \circ M = I_W \qquad (1)$$
>
> where I_V is the identity map on V, and I_W is the identity map on W. Then we call L is **invertible** and W is the **inverse** of L and write $W = L^{-1}$.

> **Fact 4:** If $L : V \to W$ is a linear function and is invertible, then its inverse L^{-1} is a also linear function.

Example. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, so that L and M are given by the matrix multiplication by A and B, respectively. Then the condition (1) is reduced to

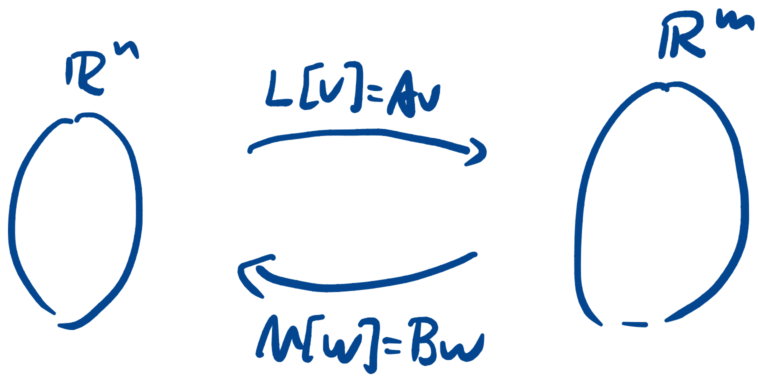
$$AB = I_m, \quad BA = I_n.$$

$L[v] = Av.$

$M[w] = Bw.$

$L \circ M = AB : \mathbb{R}^m \to \mathbb{R}^m.$

$M \circ L = BA : \mathbb{R}^n \to \mathbb{R}^n.$



Definition: Let $L : V \to W$ is a linear operator.

- Since the definition is symmetric in L and L^{-1}, if L^{-1} exists, then $(L^{-1})^{-1} = L$.
- If $M : W \to V$ satisfies $M \circ L = I_V$, then M is called a **left inverse** for L.
- If $M : W \to V$ satisfies $L \circ M = I_W$, then M is called a **right inverse** for L.

Example. Let $J[f](x) = \int_a^x f(t)dt$ be the integration operator, and $D[f](x) = f'(x)$ be differentiation.

1. Compute $D \circ J$.
2. Compute $J \circ D$.

(1) $(D \circ J)[f](x) = D[Jf(x)] = \frac{d}{dx} \int_a^x f(t)\,dt$ By Fundamental theorem of Calculus.

$D \circ J = I$. $\qquad = f(x)$

So, D is a left inverse of J. #

(2) $(J \circ D)[f](x) = \int_a^x f'(t)\,dt = f(x) - f(a)$.

So, D is a right inverse of J if $f(a) = 0$.

is NOT a right inverse of J if $f(a) \neq 0$.

7.2 Linear Transformations

Consider a linear function $L : \mathbb{R}^n \to \mathbb{R}^n$. We have known that "Every linear mapping L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication,

$$L[\mathbf{v}] = A\mathbf{v},$$

where A is an $m \times n$ matrix."

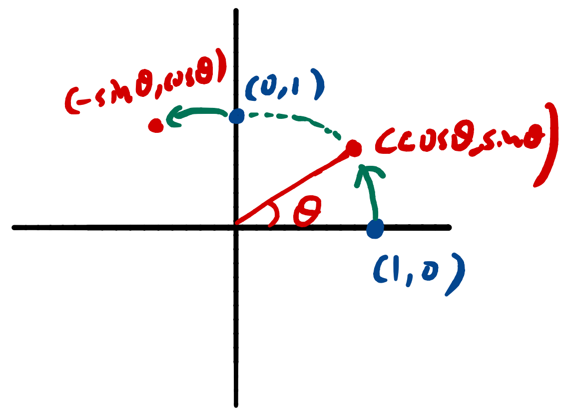
The following we will see how the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ representing the geometrical interpretation.

1. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

   $L[e_1] = Ae_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

   $L[e_2] = Ae_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

   * counterclockwise by angle θ.

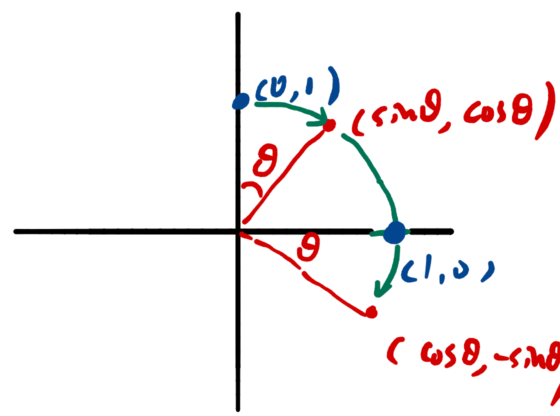


2. $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

   $L[e_1] = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$

   $L[e_2] = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$.

   * clockwise by angle θ.

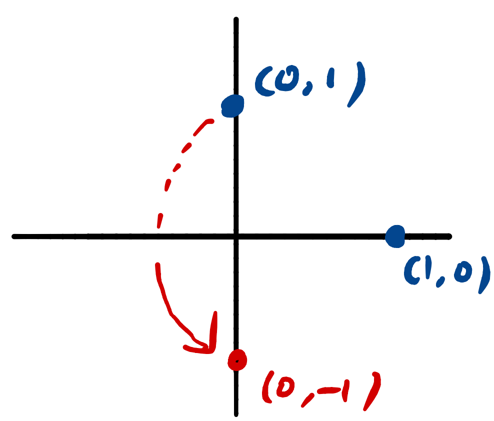


3. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

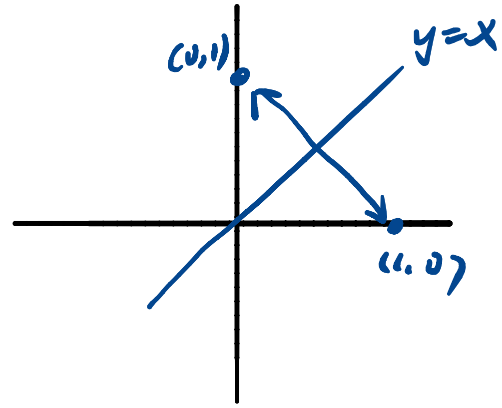
   $L[e_1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

   $L[e_2] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

   * Reflection through x-axis.



4. Reflects points through the line $x = y$:

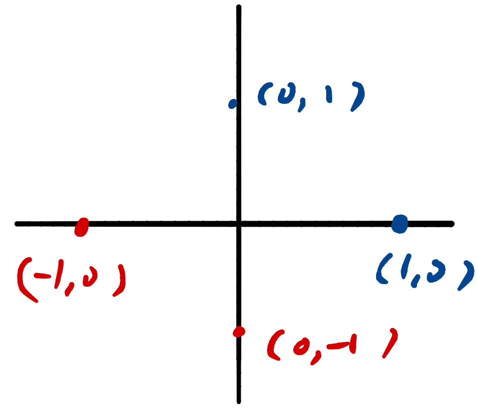


$$L\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$L\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

$$A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

5. Reflects points through the origin:



$$L[e_1] = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$L[e_2] = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}.$$

Example. Find the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which

1. first rotates points counterclockwise about the origin through $\pi/4$;
2. then reflects points through the line $x = y$.

$$L_2 x = Bx = \begin{bmatrix}\cos\pi/4 & -\sin\pi/4\\ \sin\pi/4 & \cos\pi/4\end{bmatrix} x$$

$$L_1 y = Ay = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} y.$$

$$L = L_1 \circ L_2 = AB.$$

$$= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}. \quad \#.$$