Lecture 28: Quick review from previous lecture

- Let $\mathcal{L}(V, W)$ be the set of all linear functions L mapping from vector space V to vector space W. Then $\mathcal{L}(V, W)$ is a vector space.
- If $L : V \to W$ is a linear operator and $M : W \to Z$ is another linear operator, then we can define their **composition** $M \circ L : V \to Z$ by

$$(M \circ L)[\mathbf{v}] = M[L[\mathbf{v}]].$$

- If $M : W \to V$ is an operator such that

$$\boxed{M} \circ L = I_V, \quad L \circ \boxed{M} = I_W \tag{1}$$

where I_V is the identity map on V, and I_W is the identity map on W. Then we call L is **invertible** and W is the **inverse** of L and write $W = L^{-1}$.

Today we will discuss linear transformations.

- Lecture will be recorded -

- Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas. See category "Announcements".
- Solutions for HW 5 and HW 6 are posted on Canvas. Only the solutions of these two HWs will be provided since they cannot be returned in this semester due to the closure of the campus.

§ Change of Basis

1. In $\mathbb{R}^n$, change coordinates from $\mathbf{e}_1, \cdots, \mathbf{e}_n$ to $\mathbf{v}_1, \cdots, \mathbf{v}_n$.

Consider the vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ with the coordinate in the standard basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

Ex: $\vec{x} = (2,3)$ in $\mathbb{R}^2$
$= 2e_1 + 3e_2$.
$= 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$.

that is, $\vec{\mathbf{x}} = x_1\mathbf{e}_1 + \cdots x_n\mathbf{e}_n$.

Q: How do we find the coordinate of the same vector $\vec{\mathbf{x}}$ to the new basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$?

New coord. to $\{v_1, \dots, v_n\}$.

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{e}_1,\cdots,\mathbf{e}_n} \longrightarrow \underbrace{\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}}_{\mathbf{v}_1,\cdots,\mathbf{v}_n}?$$

In other words, finding $(x_1', \cdots, x_n')^T$ such that

$$\boxed{\vec{\mathbf{x}}} = \underline{x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n} = x_1'\mathbf{v}_1 + \cdots + x_n'\mathbf{v}_n.$$

$$\vec{x} = \underbrace{[e_1\ e_2\ \cdots\ e_n]}_{I_n}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{[v_1\ v_2\ \cdots\ v_n]}_{S}\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$$

S, $\boxed{\text{invertible}}$ since $\{v_1, \dots, v_n\}$ is a basis, we have $\{v_1, \dots, v_n\}$ are linearly independent.

Thus,

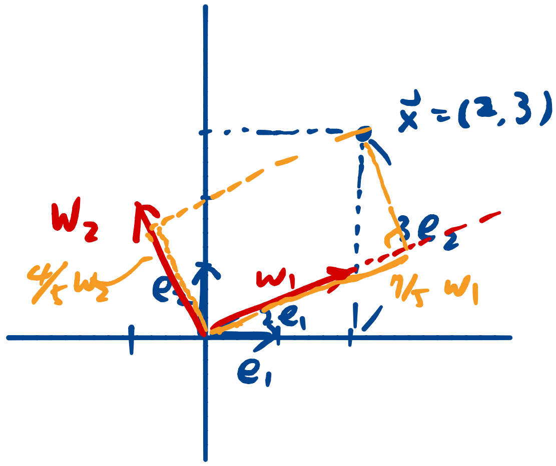
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix},$$

which implies

$$\boxed{\underbrace{\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}}_{\text{New coord. to } \{v_1,\dots,v_n\}} = S^{-1}\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\text{old coord. to } \{e_1,\dots,e_n\}}.}$$

Example. Take a point $\vec{\mathbf{x}} = (2,3)$ in $\mathbb{R}^2$, thus $\vec{\mathbf{x}} = 2\mathbf{e}_1 + 3\mathbf{e}_2$. Take another basis $\mathbf{w}_1 = (2,1)^T$ and $\mathbf{w}_2 = (-1,2)^T$ for $\mathbb{R}^2$. Find the corresponding coordinate of $\vec{\mathbf{x}}$ to this new basis $\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = S^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$



where $S = [w_1 \;\; w_2]$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

so $S^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

Now, $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 4/5 \end{pmatrix}$, new coordinate to basis $\{w_1, w_2\}$.

<u>Check</u>: $\frac{7}{5} w_1 + \frac{4}{5} w_2 = \frac{7}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \boxed{= \vec{x}}.$$

2. In $\mathbb{R}^n$, change coordinates from $\mathbf{v}_1, \cdots, \mathbf{v}_n$ to $\mathbf{w}_1, \cdots, \mathbf{w}_n$.
Consider the vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ with the coordinate in a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

that is, $\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n$.

Q: How do we find the coordinate of the same vector $\vec{\mathbf{x}}$ to the new basis $\mathbf{w}_1, \cdots, \mathbf{w}_n$?

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{v}_1,\cdots,\mathbf{v}_n} \longrightarrow \underbrace{\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}}_{\mathbf{w}_1,\cdots,\mathbf{w}_n} ?$$

In other words, finding $(x_1', \cdots, x_n')^T$ such that

$$\boxed{\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n = x_1'\mathbf{w}_1 + \cdots x_n'\mathbf{w}_n.}$$

$$\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}}_{y}$$

- S, T are invertible since $\{v_1, \ldots, v_n\}, \{w_1, \ldots, w_n\}$ are basis for $\mathbb{R}^n$.

$$S x = T y$$

$$\Rightarrow \boxed{\underbrace{y}_{\text{new coord. to basis } \{w_1,\ldots,w_n\}} = T^{-1} S \underbrace{x}_{\text{old coord. to basis } \{v_1,\ldots,v_n\}}} \quad \#$$

Example. Let $p(x) = 2x^2 + x + 1$, where $(2, 1, 1)^T$ is the coordinate of p in the monomial basis $\{x^2, x, 1\}$. Change the coordinate from the monomial basis $\{x^2, x, 1\}$ to the new basis $\{x^2 - x, x - 1, 1\}$ for $V = \mathcal{P}^{(2)}$.

$p(x) = 2x^2 + x + 1$. coefficient vector to $\{x^2, x, 1\}$

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

	coefficient vectors to $\{x^2, x, 1\}$		coefficient vector to $\{x^2, x, 1\}$
x^2	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$x^2 - x$	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = v_1$
x	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$x - 1$	$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = v_2$
1	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	1	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v_3$.

So, $S = [v_1 \ v_2 \ v_3] = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = S^{-1} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}}.$$

<u>Check</u>: $p(x) = 2x^2 + x + 1 = \underbrace{2(x^2 - x) + 3(x - 1) + 4 \cdot 1}_{\shortparallel \atop 2x^2 + x + 1.}$.

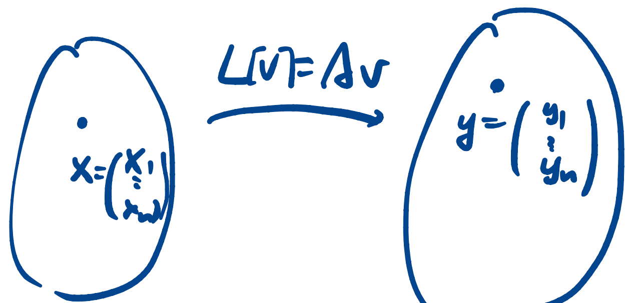
3. A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L[\mathbf{v}] = A\mathbf{v}$.

Suppose $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is basis for $\mathbb{R}^n$ and $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for $\mathbb{R}^n$.

Consider the vector $\vec{\mathbf{x}}$ in V with the coordinate in a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

that is, $\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n$.



Q: How do we find the coordinate $(y_1, \cdots, y_n)^T$ of the vector $L[\vec{\mathbf{x}}]$ to the basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$?

$$\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n \longrightarrow \underline{\vec{\mathbf{y}} = L[\vec{\mathbf{x}}]} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$$

Now, $L[\vec{x}] = L[x_1 v_1 + \cdots + x_n v_n]$

$= A[x_1 v_1 + \cdots + x_n v_n]$

$= A \underbrace{[v_1 \cdots v_n]}_{S} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{x} = ASx. \quad — (1)$

$\vec{y} = L[\vec{x}] = y_1 v_1 + \cdots + y_n v_n = \underbrace{[v_1 \cdots v_n]}_{S} \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{y}$

$= Sy \quad — (2)$

Thus, $(1) = (2)$.

$ASx = Sy.$

$\Rightarrow \quad y = \boxed{S^{-1}AS}\, x = Bx.$ "B"

Let A and B are $n \times n$ matrices. We say B **is similar to** A if there exists an invertible matrix S such that $B = S^{-1}AS$.

4. A linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$, $L[\mathbf{v}] = A\mathbf{v}$.

Suppose $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is basis for $\mathbb{R}^n$ and $\mathbf{w}_1, \cdots, \mathbf{w}_m$ is a basis for $\mathbb{R}^m$.

Consider the vector $\vec{\mathbf{x}}$ in V with the coordinate in a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

that is, $\vec{\mathbf{x}} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n$.

Q: How do we find the coordinate $(y_1, \cdots, y_m)^T$ of the vector $L[\vec{\mathbf{x}}]$ to the basis $\mathbf{w}_1, \cdots, \mathbf{w}_m$?

$$\vec{\mathbf{x}} = \underline{x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n} \longrightarrow \vec{\mathbf{y}} = L[\vec{\mathbf{x}}] = \underline{y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m}$$

Similarly,

① $L[\vec{x}] = A S x$, where $S = [v_1 \cdots v_n]$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

② $y = L[\vec{x}] = \underbrace{[w_1 \cdots w_m]}_{T} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = T y$, where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

Thus, ① = ② :

$$A S x = T y.$$

$$\Rightarrow \boxed{y = \underbrace{T^{-1} A S}_{B} x} \ \#$$

$$\boxed{y = B x.} \ \#$$

In general, we can conclude the following result.

Theorem: Suppose $L : V \to W$ is a linear operator, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V, and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ form a basis for W. We can write

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \in V, \qquad \mathbf{w} = y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m \in W,$$

where $\mathbf{x} = (x_1, \cdots, x_n)^T$ are the coordinates of $\mathbf{v}$ relative to the basis of V and $\mathbf{y} = (y_1, \cdots, y_m)^T$ are the coordinates of $\mathbf{w}$ relative to the basis of W. Then in these coordinates, the linear function

$$L[\mathbf{v}] = \mathbf{w},$$

is given by multiplication by an $m \times n$ matrix B, and then

$$B\underline{\mathbf{x}} = \mathbf{y}.$$

EX: $L[\vec{x}] = A\vec{x}$, $A \in M_{m\times n}$.

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \underbrace{T^{-1}AS}_{B} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Example. Suppose we have the operator $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L[(x, y, z)^T] = \begin{pmatrix} 1x + 1y \\ 1y + 1z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ represents L in the standard basis ($\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, $\{e_1, e_2\}$ for $\mathbb{R}^2$.

Q: Consider the basis $\mathbf{v}_1 = (1, 0, 1)^T$, $\mathbf{v}_2 = (1, -1, 0)^T$, $\mathbf{v}_3 = (0, 1, -1)^T$ of $\mathbb{R}^3$, and $\mathbf{w}_1 = (1, 1)^T$, $\mathbf{w}_2 = (-1, 1)^T$ of $\mathbb{R}^2$. What would be the matrix representation of L in these bases?

$$S = [v_1 \ v_2 \ v_3] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

$$T = [w_1 \ w_2] = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

matrix representation in basis $\{v_1, v_2, v_3\}$, and $\{w_1, w_2\}$:

$$B = T^{-1} A S = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}_{3 \times 2}$$

$v_1 = 1 v_1 + 0 v_2 + 0 v_3$

	$\{v_1, v_2, v_3\}$	$\xrightarrow{B}$	$\{w_1, w_2\}$	
v_1	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\xrightarrow{B}$	$\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$;	$1 \cdot w_1 + 0 w_2$.
v_2	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\xrightarrow{B}$	$\frac{1}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$;	$\frac{1}{2}(-w_1) + \frac{1}{2}(-w_2)$
v_3	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\xrightarrow{B}$	$\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$;	$\frac{1}{2} w_1 + \frac{1}{2}(-w_2)$.

✓ So applying B to the coefficients of a vector $\mathbf{v}$ in the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ returns the coefficients of $L[\mathbf{v}]$ in the basis $\mathbf{w}_1, \mathbf{w}_2$.

Check: $L[v_1] = A v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= 1 \cdot w_1 + 0 \, w_2$

*One reason for changing basis is that some coordinate systems are better-adapted for a particular operator L than others.

Example. Consider the operator

$$L[(x,y)^T] = \underbrace{\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}}_{} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - y \\ 5y - x \end{pmatrix}$$

'' A

Let's change basis to $\mathbf{w}_1 = (1,-1)^T$, $\mathbf{w}_2 = (1,1)^T$. The matrix representation of L in these bases is

$$\underset{\text{''}\,B}{S^{-1}AS} = [\mathbf{w}_1, \mathbf{w}_2]^{-1} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} [\mathbf{w}_1, \mathbf{w}_2]$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}}$$

$$B\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6a \\ 4b \end{pmatrix}$$

- In other words, if $\mathbf{x} = a\mathbf{w}_1 + b\mathbf{w}_2$, then $L[\mathbf{x}] = 6a\mathbf{w}_1 + 4b\mathbf{w}_2$. So L scales along the direction of $\mathbf{w}_1$ by 6, and scales along the direction of $\mathbf{w}_2$ by 4.
- The simple geometry of L is only revealed by the new basis; it is not apparent from the matrix in the standard basis, $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$.

$w_1 = 1 \cdot w_1 + 0 \cdot w_2$

$$\{w_1, w_2\} \xrightarrow{B} \{w_1, w_2\}.$$

$$w_1 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} 6 \\ 0 \end{pmatrix}; \quad \boxed{6w_1}$$

$$w_2 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} 0 \\ 4 \end{pmatrix}; \quad \boxed{4w_2}$$