Lecture 29: Quick review from previous lecture

• **Theorem:** Suppose $L : V \to W$ is a linear operator, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $V$, and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ form a basis for $W$. We can write

\[
\mathbf{v} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n \in V, \quad \mathbf{w} = y_1 \mathbf{w}_1 + \cdots + y_m \mathbf{w}_m \in W,
\]

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ are the coordinates of $\mathbf{v}$ relative to the basis of $V$ and $\mathbf{y} = (y_1, \ldots, y_m)^T$ are the coordinates of $\mathbf{w}$ relative to the basis of $W$. Then in these coordinates, the linear function

\[
L[\mathbf{v}] = \mathbf{w},
\]

is given by multiplication by an $m \times n$ matrix $B$, and then

\[
B \mathbf{x} = \mathbf{y}.
\]

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Basis

$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ \quad $\overset{L}{\rightarrow} \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$

Then the matrix representation of $L$ in these bases is

\[
B = T^{-1} A S,
\]

where $S = [\mathbf{v}_1 \ldots \mathbf{v}_n]$, $T = [\mathbf{w}_1 \ldots \mathbf{w}_m]$.

- Lecture will be recorded -

• Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas. See category ”Announcements”.

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**Example.** Let $V = \mathcal{P}^{(2)}$, the vector space of polynomials of degree $\leq 2$; $W = \mathcal{P}^{(1)}$, the vector space of polynomials of degree $\leq 1$; and

$$L[p](x) = p'(x).$$

Consider the basis $\{x^2, x, 1\}$ for $V$ and the basis for $\{x, 1\}$ for $W$. Find the matrix representation of $L$ in these bases.

$\begin{align*}
\begin{pmatrix}
p^{(2)} \\
\{x^2, x, 1\}
\end{pmatrix}
&\overset{L}{\rightarrow}
\begin{pmatrix}
p^{(1)} \\
\{x, 1\}
\end{pmatrix} \\
\begin{pmatrix}
p(x) \\
= ax^2 + bx + c
\end{pmatrix}
&\overset{L}{\rightarrow}
\begin{pmatrix}
p'(x) = 2ax + b \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\end{align*}$

$L[x^2] = 2x = \boxed{2x + 0 \cdot 1}$

$L[x] = 1 = 0x + 1 \cdot 1$

$L[1] = 0 = 0x + 0 \cdot 1$

Thus, $L$ in these bases is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

**Observation:**

$$L \begin{pmatrix} ax^2 + bx + c \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix}$$

*Note that the matrix $A$ that represents $L$ depends on the choice of basis for $V$ and $W!"
Example. Again, we consider the linear operator $L : V \to W$ defined by

$$L[p](x) = p'(x).$$

Suppose that instead of the monomial basis for $V$ and $W$, we had instead used the basis $\{x^2 - x, x - 1, 1\}$ for $V = P^{(2)}$ and $\{2x, 1\}$ for $W = P^{(1)}$. Find the matrix representation of $L$ in these bases.

From the previous example, $L[p] = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} (g)$. In these bases $\{x^2, x, 1\}$ and $\{x, 1\}$.

$$B = T^{-1} A S,$$

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

So,

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}.$$

Observation:

$$x^2 - x = \frac{1}{2} (x^2 - x) + \frac{1}{2} (x - 1) + 0.1.$$
§ Canonical Form of the operator $L$.

We take any matrix $A = A_{m \times n}$. Suppose the rank of $A$ is $r$. Let $L[x] = Ax$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for $\text{coimg } A$, and $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ a basis for $\text{ker } A$, the orthogonal complement. So $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis for $\mathbb{R}^n$.

As we’ve seen,

$$\{\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_r = A\mathbf{v}_r\}$$

is a basis for $\text{img } A$.

Take $\mathbf{w}_{r+1}, \ldots, \mathbf{w}_m$ to be any basis for $\text{coker } A$. Then

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{w}_{r+1}, \ldots \mathbf{w}_m\}$$

is a basis for $\mathbb{R}^m$.

Q: What is the matrix for $L$ in the bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ of $\mathbb{R}^m$?

$$\begin{array}{c}
\mathbb{R}^n \\
\downarrow A \\
\mathbb{R}^m \\
\end{array}$$

\[ B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{r \times (n-r)} \times \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{(n-r) \times m} \\
= \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}_{m \times n}.
\]

* In other words, the top $r$-by-$r$ block is the identity, and everywhere else it has 0. This is the **canonical form** of the operator $L$, that depends “only” on its rank.
Example. Let the operator \( L[x] = Ax \), where
\[
A = \begin{pmatrix}
1 & 2 \\
2 & 4 \\
-1 & -2
\end{pmatrix}.
\]

Find the canonical form of the operator \( L[x] = Ax \).

Find basis for \( \text{rang } A : A \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \)
\( \text{rang } A = \text{span} \{ (2, \_1) \} \)

Find basis for \( \ker A : \{ v_2 = (-2, _) \} \)

Find basis for \( \text{coker } A : \{ (-2, _0), (_0, _) \} \)

Basis for \( \text{img } A = \{ Av_1 = (15) \} \)
\( \tilde{w}_1 = \begin{pmatrix} 15 \\ -5 \end{pmatrix} \)

Now, we have \( S = [v_1, v_2] \), \( T = [w_1, w_2, w_3] \)
\( B = T^{-1} A S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)

\( \therefore \) we can also find \( B \) by
\[
A[v_1] = 1w_1 = 1w_1 + 0w_2 + 0w_3
\]
\( A[v_2] = 0 = 0w_1 + ow_2 + ow_3 \)

Thus, \( B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)

Remark: We can rewrite \( A = TBS^{-1} \)
\( \therefore \) we can "factorize \( A \)" into \( TBS^{-1} \).
Chapter 8 Eigenvalues and Singular Values

We will discuss 8.2, 8.3, 8.5, and 8.7.

8.2 Eigenvalues and Eigenvectors

As we will see, eigenvectors are a natural basis for expressing the action of symmetric linear operators.

**Definition:** If \( A = A_{n \times n} \) is a square matrix, we say that a scalar \( \lambda \) is an **eigenvalue** of \( A \) if there is a non-zero vector \( \mathbf{v} \neq \mathbf{0} \) satisfying

\[
A \mathbf{v} = \lambda \mathbf{v}
\]

If \( \lambda \) is an eigenvalue, we say a vector \( \mathbf{v} \neq \mathbf{0} \) satisfying \( A \mathbf{v} = \lambda \mathbf{v} \) is an **eigenvector**.

*Important:* The zero vector \( \mathbf{0} \) is **not** allowed to be an eigenvector, by definition.

**Properties:**

- In geometric terms, the eigenvectors of \( A \) are those vectors that are stretched/scaled by \( A \).
- The eigenvalue \( \lambda \) is the amount by which the eigenvector \( \mathbf{v} \) is stretched.
- Note that even though \( \mathbf{v} \neq \mathbf{0} \), we may have \( \lambda = 0 \).

\[
\text{When } \lambda = 0, \quad A \mathbf{v} = \mathbf{0} \quad \text{for } \mathbf{v} \in \ker A.
\]

Thus it can happen if \( \mathbf{0} \in \ker A \).
§ How to find eigenvalues and eigenvectors. Let’s rewrite the equations $A \mathbf{v} = \lambda \mathbf{v}$ into

\[(A - \lambda I) \mathbf{v} = \mathbf{0},\]  

where $I$ is the identity matrix.

Clearly, it is a homogeneous linear system, and thus $\mathbf{v} = \mathbf{0}$ is a solution of (1).

**Q:** How to find its nonzero solutions (eigenvectors $\mathbf{v}$)?

In other words, the eigenvectors $\mathbf{v}$ with eigenvalue $\lambda$ are the non-zero vectors in the kernel of $A - \lambda I$.

Thus, we have the following fact.

**Fact 1:** A scalar $\lambda$ is an eigenvalue of $n \times n$ matrix $A$ if and only if $A - \lambda I$ is singular ($\text{rank}(A - \lambda I) < n$).

From Fact 1, we immediately have

**Fact 2:** A scalar $\lambda$ is an eigenvalue of $n \times n$ matrix $A$ if and only if $\lambda$ is a solution to the characteristic equation

\[\det(A - \lambda I) = 0.\]

We define $p_A(\lambda) = \det(A - \lambda I)$, the eigenvalues of $A$ are the roots of $p_A(\lambda)$, i.e. the values $\lambda$ at which $p_A(\lambda) = 0$. 
Example. Find eigenvalues and eigenvectors. Let \( A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \).

1.) \[ 0 = \det (A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ -2 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = (\lambda+1)(\lambda-2). \]

So, \( \lambda = -1, \ 2 \). eigen values.

2.) Find eigenvectors. (Find ker \( (A-\lambda I) \))

\( \lambda = -1 \):

\[ A - \lambda I = A + I = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ A \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\( \lambda = 2 \):

\[ A - 2 I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Remark: If \( \mathbf{v} \) is an eigenvector \( A \) for the eigenvalue \( \lambda \), then so is every nonzero scalar multiple of \( \mathbf{v} \), that is, \( c \mathbf{v} \) for scalar \( c \neq 0 \).