Lecture 29: Quick review from previous lecture

- **Theorem:** Suppose $L : V \to W$ is a linear operator, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V, and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ form a basis for W. We can write

$$\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \in V, \qquad \mathbf{w} = y_1\mathbf{w}_1 + \cdots + y_m\mathbf{w}_m \in W,$$

where $\mathbf{x} = (x_1, \cdots, x_n)^T$ are the coordinates of $\mathbf{v}$ relative to the basis of V and $\mathbf{y} = (y_1, \cdots, y_m)^T$ are the coordinates of $\mathbf{w}$ relative to the basis of W. Then in these coordinates, the linear function

$$L[\mathbf{v}] = \mathbf{w},$$

is given by multiplication by an $m \times n$ matrix B, and then

$$\boxed{B}\mathbf{x} = \mathbf{y}.$$

Today we will discuss eigenvalues and eigenfunctions.

Basis

$\{v_1, v_2, \ldots, v_n\} \xrightarrow{L[v] = Av} \{w_1, \ldots, w_m\}$

Then the matrix representation of L in these bases is

$$B = T^{-1} A S,$$

where $S = [v_1 \ldots v_n]$, $T = [w_1 \ldots w_m]$.

- Lecture will be recorded -

- Midterm 2 will cover 2.5, Chapter 3, and 4.1 - 4.4. Details about Midterm 2 has been announced on Canvas. See category "Announcements".

Example. Let $V = \mathcal{P}^{(2)}$, the vector space of polynomials of degree ≤ 2; $W = \mathcal{P}^{(1)}$, the vector space of polynomials of degree ≤ 1; and

$$L[p](x) = p'(x).$$

Consider the basis $\{x^2, x, 1\}$ for V and the basis for $\{x, 1\}$ for W. Find the matrix representation of L in these bases.

$$\mathcal{P}^{(2)} \xrightarrow{L} \mathcal{P}^{(1)}$$

$$\{x^2, x, 1\} \qquad \{x, 1\}$$

$p(x) = ax^2 + bx + c$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \xrightarrow{L} p'(x) = \boxed{2a}x + \boxed{b}, \begin{pmatrix} 2a \\ b \end{pmatrix}$$

$$L[x^2] = 2x = 2\,x + 0 \cdot 1$$

$$L[x] = 1 = 0\,x + 1 \cdot 1$$

$$L[1] = 0 = 0\,x + 0 \cdot 1$$

Thus, L in these bases is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Observation:

$$L[ax^2 + bx + c] = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b \end{pmatrix}!$$

*

*Note that the matrix A that represents L depends on the choice of basis for V and W!

Example. Again, we consider the linear operator $L : V \to W$ defined by

$$L[p](x) = p'(x).$$

Suppose that instead of the monomial basis for V and W, we had instead used the basis $\{x^2 - x, x - 1, 1\}$ for $V = \mathcal{P}^{(2)}$ and $\{2x, 1\}$ for $W = \mathcal{P}^{(1)}$. Find the matrix representation of L in these bases.

$$\underbrace{\{x^2-x,\ x-1,\ 1\}}_{P^{(2)}} \xrightarrow{L} \underbrace{\{2x,\ 1\}}_{P^{(1)}}$$

From the previous example, $L[p] = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
in these bases $\{x^2, x, 1\}$ and $\{x, 1\}$.

$\boxed{B = T^{-1} A S}$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

$2x = 2(x) + 0(1)$

$1 = 0x + 1 \cdot 1$

$x^2 - x = 1\,x^2 + (-1)x + 0 \cdot 1$

$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$x - 1 = 0x^2 + 1(x) - 1 \cdot 1$

So, $$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}. \quad \#$$

Observation:

$x^2 - x = 1(x^2-x) + 0(x-1) + 0 \cdot 1$

$$\{x^2-x,\ x-1,\ 1\} \xrightarrow{L} \{2x, 1\}.$$

$x^2 - x$ $\quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{B} B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $2x - 1(1)$

$x - 1$ $\quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{B} B\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $0(2x) + 1(1)$

§ Canonical Form of the operator L.

We take any matrix $A = A_{m \times n}$. Suppose the rank of A is r. Let $L[\mathbf{x}] = A\mathbf{x}$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for coimg A, and $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ a basis for ker A, the orthogonal complement. So $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis for $\mathbb{R}^n$.

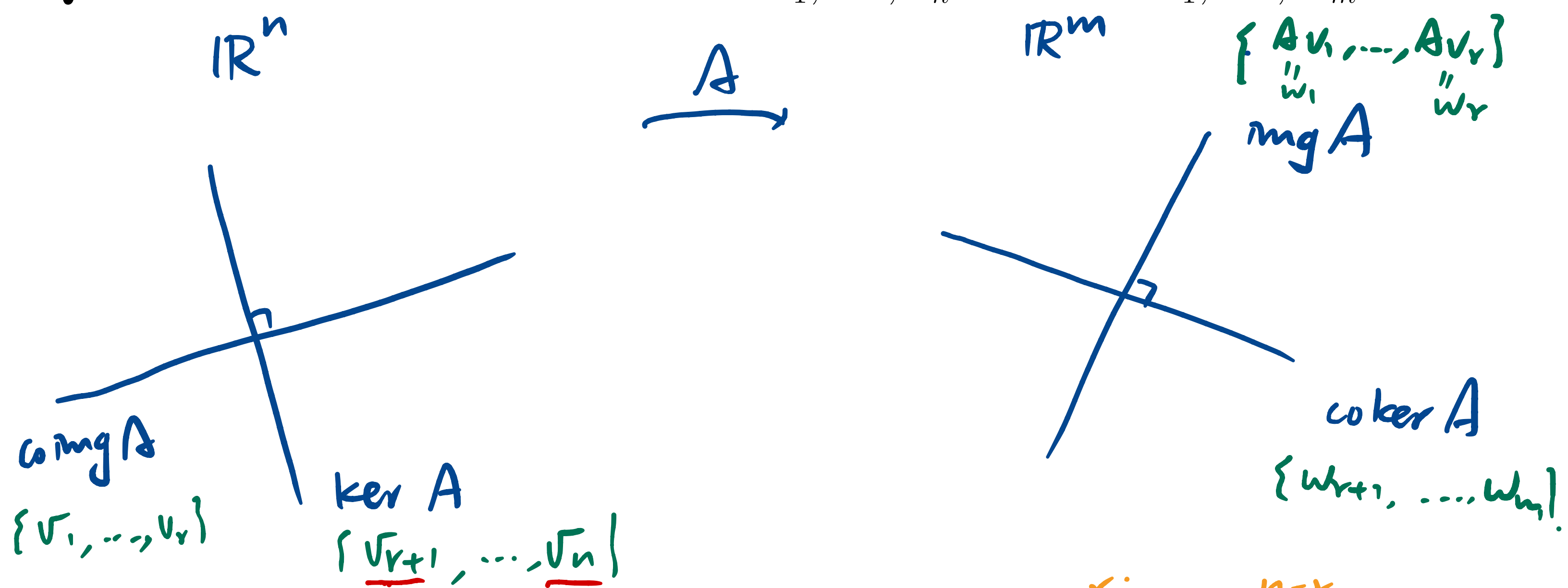
As we've seen,

$$\{\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_r = A\mathbf{v}_r\} \qquad \text{is a basis for img } A.$$

Take $\mathbf{w}_{r+1}, \ldots, \mathbf{w}_m$ to be any basis for coker A. Then

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{w}_{r+1}, \cdots \mathbf{w}_m\} \qquad \text{is a basis for } \mathbb{R}^m.$$

Q: What is the matrix for L in the bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ of $\mathbb{R}^m$?



① $L[v_i] = Av_i = 1w_i$, $1 \le i \le r$

$L[v_j] = Av_j = 0$, $j > r$.

Ex: $L[v_1] = 1w_1 + 0w_2 + \ldots + 0w_m$

$L[v_2] = 0w_1 + 1w_2 + \ldots + 0w_m$

$$B = \begin{array}{c} \\ r \\ \\ \\ \end{array} \overset{\qquad r \qquad\qquad n-r}{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}}_{m \times n}.$$

Or ② $S = [v_1 \cdots v_n]$, $T = [w_1 \cdots w_m]$. $B = T^{-1}AS = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$. #

* In other words, the top r-by-r block is the identity, and everywhere else it has 0. This is the **canonical form** of the operator L, that depends "only" on its rank.

Example. Let the operator $L[\mathbf{x}] = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}.$$

Find the canonical form of the operator $L[\mathbf{x}] = A\mathbf{x}$.

Find basis for coimg A: $A \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

coimg A = span $\left\{ \underset{v_1}{\underset{\shortparallel}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}} \right\}$

Find basis for $\underline{\ker A}$: $\left\{ v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$.

Find basis for $\underline{\text{coker} A}$: $\left\{ \underset{w_2}{\underset{\shortparallel}{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}}, \underset{w_3}{\underset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}} \right\}$.

Basis for img A = $\left\{ \underset{w_1}{\underset{\shortparallel}{Av_1}} = \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} \right\}$

Now, we have $S = [v_1 \ v_2]$, $T = [w_1 \ w_2 \ w_3]$.

$$B = T^{-1} A S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{3\times 2}.$$

$\underline{\underline{\text{or}}}$ we can also find B by

$A[v_1] = 1 w_1 = 1\, w_1 + 0\, w_2 + 0\, w_3$

$A[v_2] = 0 \quad = 0\, w_1 + 0 w_2 + 0 w_3$.

Thus, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ #

$\underline{\text{Remark}}$: We can rewrite $A = TBS^{-1}$

So we can "factorize A" into TBS^{-1}. #.

Chapter 8 Eigenvalues and Singular Values

We will discuss 8.2, 8.3, 8.5, and 8.7.

8.2 Eigenvalues and Eigenvectors

As we will see, eigenvectors are a natural basis for expressing the action of symmetric linear operators.

Definition: If $A = A_{n\times n}$ is a square matrix, we say that a scalar λ is an **eigenvalue** of A if there is a non-zero vector $\mathbf{v} \neq \mathbf{0}$ satisfying

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}}$$

If λ is an eigenvalue, we say a vector $\mathbf{v} \neq \mathbf{0}$ satisfying $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector**.

***Important:** The zero vector $\mathbf{0}$ is **not** allowed to be an eigenvector, by definition.

$$v \xrightarrow{A} \lambda v$$

Properties:

- In geometric terms, the eigenvectors of A are those vectors that are stretched/scaled by A.
- The eigenvalue λ is the amount by which the eigenvector $\mathbf{v}$ is stretched.
- Note that even though $\mathbf{v} \neq \mathbf{0}$, we may have $\lambda = 0$.

When $\lambda = 0$, $A \overset{\neq 0}{v} = 0 v = 0$.

Thus it can happen if $\underset{\neq 0}{v} \in \ker A$.

§ **How to find eigenvalues and eigenvectors.** Let's rewrite the equations $A\mathbf{v} = \lambda\mathbf{v}$ into

$$\boxed{(A - \lambda I)\mathbf{v} = \mathbf{0},} \tag{1}$$

where I is the identity matrix.

Clearly, it is a homogeneous linear system, and thus $\mathbf{v} = \mathbf{0}$ is a solution of (1).

Q: How to find its <u>nonzero solutions</u> (eigenvectors $\mathbf{v}$)?

In other words, the eigenvectors $\mathbf{v}$ $\neq 0$ with eigenvalue λ are the non-zero vectors in the kernel of $A - \lambda I$.

$A - \lambda I$ is NOT full rank.

Thus, we have the following fact.

Fact 1: A scalar λ is an eigenvalue of $n \times n$ matrix A if and only if $A - \lambda I$ is singular (~~rank$A < n$~~).

rank$(A - \lambda I) < n$.

From Fact 1, we immediately have

Fact 2: A scalar λ is an eigenvalue of $n \times n$ matrix A if and only if λ is a solution to the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

We define

$$p_A(\lambda) = \det(A - \lambda I),$$

polynomial of degree n.

the eigenvalues of A are the **roots** of $p_A(\lambda)$, i.e. the values λ at which $p_A(\lambda) = 0$.

Example. Find eigenvalues and eigenvectors. Let $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$.

(1) $0 = \det(A - \lambda I)$

$= \det \begin{pmatrix} 1-\lambda & -1 \\ -2 & -\lambda \end{pmatrix}$

$= (1-\lambda)(-\lambda) - 2 = (\lambda+1)(\lambda-2)$.

So, $\lambda = -1, 2$. eigenvalues.

(2) Find eigenvectors. (Find $\ker(A - \lambda I)$).

$\underline{\lambda = -1}$: $A - \lambda I = A + I = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$.

$\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$A \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$. $\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$. #

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$.

$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$. #

Remark: If $\mathbf{v}$ is an eigenvector A for the eigenvalue λ, then so is every nonzero scalar multiple of $\mathbf{v}$, that is, $c\mathbf{v}$ for scalar $c \neq 0$.