Lecture 3: Quick review from previous lecture

- We have learned how to use Gaussian elimination to solve a linear system $A\mathbf{x} = \mathbf{b}$ when A is **regular**, that means that a matrix only has nonzero pivots.
- We have shown that such regular matrix A can be factored as

$$A = LU,$$

where U is upper triangular and L is lower triangular.
Furthermore, L has 1's on its main diagonal, and U has non-zero elements on its main diagonal (the pivots of A).

Today we will discuss how to deal with the case when matrix A has <u>zero</u> pivot.

- Quiz 1 will be given in the **beginning** of class on Wednesday (1/29). It will cover sec. 1.1-1.3.

$10 \sim 15$ mins.

Example: Find LU factorization of the matrix

$$U=\begin{pmatrix} 5 & -10 & 5 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{②-\frac{4}{5}①} \begin{pmatrix} 5 & -10 & 5 \\ 0 & 3 & -1 \\ 1 & -1 & -2 \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 & 0 \\ \boxed{\frac{4}{5}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

undo the effect of the row operation

$$\xrightarrow{③-\frac{1}{5}①} \begin{pmatrix} 5 & -10 & 5 \\ 0 & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{pmatrix},$$

$$\xrightarrow{③-\frac{1}{3}②} \begin{pmatrix} 5 & -10 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & \underbrace{\frac{-8}{3}}_{=-3+\frac{1}{3}} \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{pmatrix}.$$

↳ upper triangular,

$$U=\begin{pmatrix} 5 & -10 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & -8/3 \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{pmatrix}.$$

Thus, $A=LU$. #

§ Use "LU factorization" to solve a linear system $A\mathbf{x} = \mathbf{b}$. Suppose $A = LU$.

$\Rightarrow L\underbrace{(Ux)}_{y} = b$.

We do this by solving 2 linear systems:

(1) $Ly = b$ (Solve for y by using "forward substitution")

$[\Delta 0]\, y = b$

(2) $Ux = y$ (Solve for x by using "back substitution")

$[{}_0\nabla]\, x = y$

Back to Example: Let

$$A = \begin{pmatrix} 5 & -10 & 5 \\ 4 & -5 & 3 \\ 1 & -1 & -2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}.$$

Solve $A\mathbf{x} = \mathbf{b}$ by using LU factorization.

First, we solve a linear system $Ly = b$.

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}.$$

Solving y by "forward subs."

$y_1 = 5$, $\quad \frac{4}{5} y_1 + y_2 = 2$, $\quad y_2 = 2 - \frac{4}{5} y_1 = 2 - 4 = -2$

$\frac{1}{5} y_1 + \frac{1}{3} y_2 + y_3 = 4$, $\quad y_3 = 4 - \frac{1}{5} y_1 - \frac{1}{3} y_2 = 4 - 1 + \frac{2}{3}$

$= 11/3$

Next, solve $Ux = y$.

$$\begin{bmatrix} 5 & -10 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 11/3 \end{pmatrix}$$

Using back substitution, $x_1 = 1/8$, $x_2 = -9/8$,

$x_3 = -11/8$. #

1.4 Pivoting and Permutations

From the following example, we will learn how to handle the situation, where some pivot of the matrix A is zero when we perform Gaussian Elimination.

Example. Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 3 & 1 & 5 \end{array}\right)$$

$$\xrightarrow[\text{③} - \text{①}]{\text{②} - 2\text{①}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{0} & -4 & 2 \\ 0 & 1 & -2 & 4 \end{array}\right)$$

zero pivot

$$\xrightarrow[\text{②}\ \text{③}]{\text{switch}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -4 & 2 \end{array}\right)$$

upper triangular, so we have finished elimination.

By using back substitution,

$$x_3 = -1/2, \quad x_2 = 3, \quad x_1 = -7/2$$

- The operation of permuting two rows of the matrix (or equivalently permuting the order of equations), is called **pivoting**.

- Pivoting is the the 2^{nd} type of elementary row operation; the 1^{st} type is to add a multiple of one row to another row.

  So now when we refer to **"Elementary Row Operation"**: It includes

  - Adding/subtracting a multiple of one row to another row;
  - Pivoting.

- If a (n×n) square matrix can be reduced to upper triangular form with all non-zero diagonal elements using elementary row operations, we say that this matrix is **nonsingular**.

  If A is square and nonsingular, then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any right hand side $\mathbf{b}$, and

  the solution $\mathbf{x}$ can be found by Gaussian elimination with pivoting.

- Every regular square matrix A is nonsingular, but the converse implication is not true.

- A matrix that is not nonsingular is called **singular**. If A is singular, not all linear systems $A\mathbf{x} = \mathbf{b}$ will have unique solutions $\mathbf{x}$; existence/uniqueness will depend on the choice of right hand side $\mathbf{b}$.

A **permutation matrix** is a matrix obtained from the identity matrix by any combination of row interchanges.

Example: Write down the 4-by-4 permutation matrix that swaps the order of rows 2 and 4.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- If B is any 4-by-k matrix, then PB is equal to B but with rows 2 and 4 permuted. For example, let

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$PB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \\ 5 & 6 \\ 3 & 4 \end{pmatrix}.$$

The same representation works for any permutation, not just row swaps.

Example: We can cyclically permute the rows, sending row 1 to row 2, row 2 to row 3, row 3 to row 4, and row 4 to row 1.

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Applying this matrix to a 4-by-k matrix B performs this permutation of the rows of B.

$$PB = \begin{pmatrix} 7 & 8 \\ 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

✓ However, in this case P is not considered an "elementary matrix", since this permutation is not simply swapping two rows.

Remark:

- Multiply two or more permutation matrices, we obtain another permutation matrix. EX. If P_1, P_2 are permutation matrices, so is $P_1 P_2$.

§ The permuted LU factorization

Every nonsingular matrix A can be reduced to upper triangular matrix by elementary row operator of type 1 and type 2.

$$PA = LU$$

- Since PA is regular (has all nonzero pivot), we can find its LU factorization.

Theorem. *A is square matrix. The following are equivalent:*

1. *A is nonsingular.*
2. *A has n nonzero pivots (either by elementary row operator type 1 or type 2 pivoting)*
3. *A has a permuted LU factorization: $PA = LU$.*

We illustrate the general method to construct LU factorization of a matrix A by doing the following example: We will systematically build L, U and P.

Example. The matrix

$$A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & -3 \\ -2 & -6 & -2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$U = A, \quad L = I_4, \quad P = I_4$

$$\xrightarrow[\text{④} - \text{①}]{\text{②} - 2\text{①},\ \text{③} + 2\text{①}} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad P = I_4.$$

$$\xrightarrow[\text{②}\ \text{④}]{\text{switch}} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{③}\ \text{④}]{\text{switch}} U = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus, $PA = LU$.