Lecture 30: Quick review from previous lecture

- Let $A = A_{n\times n}$ be a square matrix, we say that a scalar λ is an **eigenvalue** of A if there is a non-zero vector $\mathbf{v} \neq \mathbf{0}$ satisfying

$$A\mathbf{v} = \lambda \mathbf{v},$$

$v \xrightarrow{A} Av = \lambda v.$

$(A - \lambda I) v = 0.$

$v \in \ker(A - \lambda I).$

$v \neq 0$

  where $\mathbf{v}$ is called the corresponding **eigenvector**.

- A scalar λ is an eigenvalue of $n \times n$ matrix A
  $\Leftrightarrow$ $A - \lambda I$ is singular ($\cancel{\text{rank} A < n}$) $\text{rank}(A - \lambda I) < n$
  $\Leftrightarrow$ **characteristic equation** $\det(A - \lambda I) = 0$.

Today we will discuss eigenvalues and eigenfunctions.

- Lecture will be recorded -

Example. Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}.$$

Find its eigenvalues and eigenvectors.

1) $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 5-\lambda & -1 \\ 0 & -1 & 5-\lambda \end{pmatrix}.$$

$$\xrightarrow{③ + \frac{1}{5-\lambda}②} \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 5-\lambda & -1 \\ 0 & 0 & (5-\lambda - \frac{1}{5-\lambda}) \end{pmatrix}$$

$$0 = \det(A - \lambda I) = (2-\lambda)(5-\lambda)\left(5-\lambda - \frac{1}{5-\lambda}\right)$$

$$= (2-\lambda)(4-\lambda)(6-\lambda).$$

We get eigenvalues $\lambda = 2, 4, 6$.

2) Find eigenvectors: $(\ker(A - \lambda I))$

① $\lambda = 2$. $A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$(A - 2I)v_1 = 0$. $\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

② $\lambda = 4$. $A - 4I = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

$(A - 4I)v_2 = 0$. $\Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

③ $\lambda = 6$, $A - 6I = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Not every matrix needs to have real eigenvalues. For example, consider the rotation matrix:

$i = \sqrt{-1}, \quad i^2 = \sqrt{-1}^2 = -1.$

Example. $Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, if $\theta \neq k\pi$ where k is an integer. Find its eigenvalues and eigenvectors.

1) $0 = \det(Q_\theta - \lambda I) = \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$

$= (\cos\theta - \lambda)^2 + \sin^2\theta.$

$= 1 - (2\cos\theta)\lambda + \lambda^2.$

using $\sin^2\theta + \cos^2\theta = 1$

So, $\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\underbrace{\sin\theta}_{\neq 0\ (\theta \neq k\pi)}.$

$\boxed{= e^{\pm i\theta}}$

Since $\theta \neq k\pi$, λ are complex number.

2) Find eigenvectors. $\mathrm{Ker}(Q_\theta - \lambda I)$.

$Q_\theta - e^{\pm i\theta} I = \begin{pmatrix} \cos\theta - (\cos\theta \pm i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta \pm i\sin\theta) \end{pmatrix}$

$= \begin{pmatrix} \mp i\sin\theta & -\sin\theta \\ \sin\theta & \mp i\sin\theta \end{pmatrix}.$

$= -\underbrace{\sin\theta}_{\neq 0} \begin{pmatrix} \pm i & 1 \\ -1 & \pm i \end{pmatrix}.$

Since $\sin\theta \neq 0$, to find $\mathrm{Ker}(Q_\theta - e^{\pm\theta i} I)$

it is sufficient to solve $\begin{pmatrix} \pm i & 1 \\ -1 & \pm i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

[Example Continue:]

① $\underline{e^{i\theta}:}$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \xrightarrow{②-i①} \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

$$-i① = -i(i \quad 1) = (-i^2 \quad -i) = (1, -i).$$

$$\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.}$$

② $\underline{e^{-i\theta}:}$

$$\underbrace{\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$A \xrightarrow{②+i①} \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\begin{pmatrix} -i \\ 1 \end{pmatrix}.}$

Fact: If A is a real matrix with a complex eigenvalue $\lambda = a + ib$ and corresponding complex eigenvector $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, then its complex conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue with complex conjugate eigenvector $\overline{v} = \mathbf{x} - i\mathbf{y}$.

Fact: Matrix A is singular if and only if A has a zero eigenvalue.

[To see this]

If A is singular, then there exists $v \neq 0$, $v \in \operatorname{Ker} A$, that is, $Av = 0 = 0v$

eigenvalue

Thus, 0 is eigenvalue.

Fact: A and A^T have the same eigenvalues.

*However, the eigenvectors do not need to be the same.

[To see this:]

Recall: Find eigenvalues of A is to solve

$$
\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \det(A - \lambda I)^T \\
&= \underline{\det(A^T - \lambda I)}
\end{aligned}
$$

$\det B = \det B^T$

Thus, $0 = \det(A^T - \lambda I) = \det(A - \lambda I)$.

so, A, A^T have the same eigenvalues.

§ Basic Properties of Eigenvalues.

$\det(A - \lambda I) = 0.$

Let A is an $n \times n$ matrix. Recall that its **characteristic polynomial**

$$p_A(\lambda) = \det(A - \lambda I) = c_n\lambda^n + \cdots + c_1\lambda + c_0$$

is a degree n polynomial, whose roots are the eigenvalues of A.

We can in principle factor p_A in the form

$$p_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

We say that the eigenvalue λ_j has **multiplicity k** if it appears k times in the factorization of $p_A(\lambda)$.

Fact: The sum of all the eigenvalues of a matrix **with multiplicity** (i.e. repeating each eigenvalue the number of times it appears in the factorization of $p_A(\lambda)$) equals the trace of A.

In other words, if $A = (a_{ij})$ and $p_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, then

$$(1) \qquad \boxed{\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}} = \sum_{i=1}^{n} \lambda_i.$$

Furthermore, the product of all the eigenvalues **with multiplicity** equals the determinant of A:

$$(2) \qquad \det A = \prod_{i=1}^{n} \lambda_i = \lambda_1\lambda_2\cdots\lambda_n.$$

Let's check these formulas for the 3-by-3 matrix:

Example. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Check (1) $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$. (2) $\det A = \lambda_1\lambda_2\cdots\lambda_n$.

Find eigenvalues: $\det(A - \lambda I) = 0.$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

[Example Continue]

$$\xrightarrow{(2) - \frac{1}{3-\lambda}(1)} \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & (3-\lambda)-\frac{1}{3-\lambda} & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}.$$

$$0 = \det(A-\lambda I) = (3-\lambda)\left(3-\lambda-\frac{1}{3-\lambda}\right)(2-\lambda) = (2-\lambda)^2(4-\lambda).$$

So, $\lambda = 2, 2, 4$ (eigenvalue 2 has "multiplicity 2")
(eigenvalue 4 ~ 1")

1) $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 4 = 8.$

$tr(A) = a_{11} + a_{22} + a_{33} = 3 + 3 + 2 = 8.$

Then $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$

2) $\lambda_1 \lambda_2 \lambda_3 = 2 \cdot 2 \cdot 4 = 16.$

$\det A = \det \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$ $\quad A \xrightarrow{(2) - \frac{1}{3}(1)} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3-\frac{1}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\det A = 3\left(3-\frac{1}{3}\right) \cdot 2 = 16.$

Find eigenvectors: $\lambda = 2$, Finding $\ker(A-2I)$

$A - 2I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ $\quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$

General solutions for $\ker(A-2I) = \left\{ \begin{pmatrix} -y \\ y \\ z \end{pmatrix} \right\}$

Its basis $\left\{ \underset{v_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}.$ (v_1, v_2 eigenvectors),

$\lambda = 4$. $A - 4I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$ $\quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

8.3 Eigenvector Bases

Fact: Eigenvectors corresponding to different eigenvalues are linearly independent.

More generally, if $\lambda_1, \ldots, \lambda_k$ are pairwise distinct eigenvalues of A, then the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

[To see this:] Suppose $\lambda_1 \neq \lambda_2$ are eigenvalues of A, with eigenvectors v_1, v_2

$[\, Av_1 = \lambda_1 v_1 \,, \quad Av_2 = \lambda_2 v_2 \,]$

$a_1 v_1 + a_2 v_2 = 0$, $\quad a_1, a_2$: scalars

$(A-\lambda_1 I)(a_1 v_1 + a_2 v_2) = (A-\lambda_1 I)\, 0$

Then $\underbrace{(A-\lambda_1 I)(a_1 v_1)}_{0 \text{ (since } (Av_1=\lambda_1 v_1))} + (A-\lambda_1 I)(a_2 v_2) = 0$

$a_2(\underbrace{\lambda_2 - \lambda_1}_{\neq 0})\, \underbrace{v_2}_{\neq 0} = 0 \implies a_2 = 0.$

Similarly, we can also get $a_1 = 0$. #

Thus, from the above Fact, we can derive that

Fact: If $n \times n$ real matrix A has n distinct real eigenvalues $\lambda_1, \cdots, \lambda_n$, then their corresponding real eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$.

If $n \times n$ matrix A (real or complex matrix) has n distinct complex eigenvalues $\lambda_1, \cdots, \lambda_n$, then their corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ form a basis of $\mathbb{C}^n$.

Definition: We say that an eigenvalue λ of a matrix A is **complete** if the number of linearly independent eigenvectors with eigenvalue λ is equal to the multiplicity of λ.