Lecture 31: Quick review from previous lecture

- A and A^T has the same eigenvalues.
- $\mathrm{tr}(A) \overset{(\mathrm{def})}{=} \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$. (2) $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$.
- If $\lambda_1, \ldots, \lambda_k$ are pairwise distinct eigenvalues of A, then the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.
- An eigenvalue λ of a matrix A is **complete** if the number of linearly independent (eigenvectors) with eigenvalue λ is equal to the multiplicity of λ.

Today we will discuss Diagonalization.

- Lecture will be recorded -

- Solutions for Midterm 2 has been posted on Canvas, see "Announcements".

Example.

- $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$ has only one eigenvalue c with multiplicity 2, with only one eigenvector $(1,0)^T$. Its eigenvalue c is NOT a complete eigenvalue.

$0 = \det(A - \lambda I) = \det\begin{pmatrix} c-\lambda & 1 \\ 0 & c-\lambda \end{pmatrix} = (c-\lambda)^2$

$A - cI = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\ker(A - cI) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = V_c$, eigenspace

$\dim V_\lambda = 1$

- On the other hand, the matrix $B = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ also has only one eigenvalue c with multiplicity 2, but it had two linearly independent eigenvectors $(1,0)^T$ and $(0,1)^T$ (and any non-zero linear combination of these). Its eigenvalue c is a complete eigenvalue.

$0 = \det(B - \lambda I) = (c-\lambda)^2$.

$B - cI = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\ker(B - cI)$ has basis $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = V_c$, eigenspace with $\dim V_c = 2$.

Definition: If A is a matrix with eigenvalue λ, we define the **eigenspace** of λ to be

$$V_\lambda = \ker(A - \lambda I).$$

Then

$\dim V_\lambda =$ the number of linearly independent eigenvectors of A with eigenvalue λ.

Thus, if $\dim V_\lambda =$ the multiplicity of λ, then λ is complete.
*It can be shown that $\dim V_\lambda$ is never greater than λ's multiplicity.

Definition: If all eigenvalues of A are complete, we say the matrix A itself is a **complete matrix**.

- $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$ is NOT complete
- $B = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ is complete.

Fact: If $n \times n$ matrix A is complete, then we can form a basis of $\mathbb{C}^n$ with its eigenvectors.

§ **Diagonalization.** Consider the linear operator $L[\mathbf{v}] = A\mathbf{v}$. Suppose matrix A is complete, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are its basis of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Q: What happens if we change basis, and represent A in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$?

[To see this]. Denote by B the matrix that represents the operator $L[\mathbf{v}] = A\mathbf{v}$ in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$. As we've seen,

$$\boxed{B = V^{-1}AV}$$

where

$$V = [\mathbf{v}_1, \ldots, \mathbf{v}_n].$$ eigenvectors of A.

$$VB = AV = A[v_1, \ldots, v_n] = [Av_1, \ldots, Av_n]$$
$$= [\lambda_1 v_1, \ldots, \lambda_n v_n]$$

If $B = [b_1, \ldots, b_n]$, then

$$VB = V[b_1, \ldots, b_n] = [Vb_1, \ldots, Vb_n].$$

Thus, $Vb_j = \lambda_j v_j$, $1 \le j \le n$.

Writing $b_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$, then

$$Vb_j = \sum_{i=1}^{n} b_{ij} v_i = \lambda_j v_j.$$

Since $v_1, \ldots, v_n$ are linearly independent, this implies $\begin{cases} b_{ij} = 0, & i \ne j \\ b_{jj} = \lambda_j \end{cases}$. It leads to $B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ ✱ □

From above, we have shown that we can factor any complete matrix A as follows: $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = V^{-1} A V$

$$A = VDV^{-1}$$

where $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the matrix of eigenvectors, and

$$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

is the diagonal matrix of eigenvalues.

✓ In other words, representing the operator $L[\mathbf{v}] = A\mathbf{v}$ in the basis of A's eigenvectors gives a diagonal matrix. (B)

Definition: We say that the matrix A is **diagonalizable**, meaning it can be factored in the form $A = VDV^{-1}$ where D is diagonal and V is nonsingular.

Fact: A matrix is complete if and only if it is diagonalizable.

Let's revisit the examples. Lecture 30

Example. $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. We have found its eigenvalues $\lambda = 2, 2, 4$.

Moreover,

complete

$$\text{eigenvalue } \lambda = 2, \qquad \text{eigenvectors } \mathbf{v}_1 = (-1,1,0)^T, \quad \mathbf{v}_2 = (0,0,1)^T,$$
$$\text{eigenvalue } \lambda = 4, \qquad \text{eigenvector } \mathbf{v}_3 = (1,1,0)^T.$$

complete

Thus, the matrix A is complete. Moreover,

$$\boxed{A = VDV^{-1},} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & & \\ & 2 & \\ & & 4 \end{pmatrix}\underbrace{\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}}_{\text{Ex: To compute } V^{-1}.}$$

where $D = \text{diag}(2,2,4)$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

Example. We already have found the eigenvectors and eigenvalues of the rotation matrix $Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. (in lecture 30)

We've seen that Q_θ has eigenvalues $\underbrace{e^{i\theta}}_{\lambda_1}$ and $\underbrace{e^{-i\theta}}_{\lambda_2}$, and eigenvectors $\underbrace{(i,1)^T}_{v_1}$, $\underbrace{(-i,1)^T}_{v_2}$.

$$Q_\theta = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix} \#$$

§ Some properties

- Suppose $A = VDV^{-1}$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. What is A^2 and A^k?

$$A^2 = VD\underbrace{V^{-1}V}DV^{-1} = VDIDV^{-1} = VDDV^{-1}$$
$$= VD^2V^{-1}$$

In general,

$$A^k = VD^kV^{-1}.$$
$$= [v_1 \cdots v_n]\begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}[v_1 \cdots v_n]^{-1}.$$

Fact: A^k has the same eigenvectors as A, and the eigenvalues are just $\lambda_1^k, \ldots, \lambda_n^k$.

Let $D_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $D_2 = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$

To generalized this fact:

- We say two matrices A and B are **simultaneously diagonalizable** if $A = V\boxed{D_1}V^{-1}$ and $B = V\boxed{D_2}V^{-1}$, for diagonal matrices D_1 and D_2.

$$AB = VD_1\underbrace{V^{-1}V}D_2V^{-1} = VD_1D_2V^{-1}$$
$$= V\begin{pmatrix} \lambda_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{pmatrix}V^{-1}.$$

Also, $BA = V\begin{pmatrix} \lambda_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{pmatrix}V^{-1}.$

Thus, $AB = BA$. #

Fact: If A and B are **simultaneously diagonalizable.** Then A and B have the same eigenvectors. And AB does too and the eigenvalues are just the products of the eigenvalues of A and B.

- If $A = VDV^{-1}$, then A is invertible if and only if $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ has all nonzero diagonal elements.

  Indeed,

$$\begin{aligned} \det A &= \det(VDV^{-1}) \\ &= \det V \cdot \det D \cdot \det(V^{-1}) \\ &= \det D = \lambda_1 \lambda_2 \cdots \lambda_n. \end{aligned}$$

  So $\det A \neq 0$ if and only if all $\lambda_i \neq 0$.

- Find A^{-1}.

$$\begin{aligned} A^{-1} &= (VDV^{-1})^{-1} \\ &= (V^{-1})^{-1} D^{-1} V^{-1} \\ &= V D^{-1} V^{-1} \\ &= V \begin{pmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{pmatrix} V^{-1} \end{aligned}$$

So, $\frac{1}{\lambda_j}$ is eigenvalue of A^{-1}.

§ Systems of Differential Equations.

Consider the system of differential equations

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + 4x_2 + 2x_3 \\ x_3' &= -x_1 - x_2 + x_3, \end{aligned}$$

where $x_i = x_i(t)$ is a differentiable real-valued function of the real variable t.

Clearly, $x_i(t) = 0$ is the solution of the system.

— To find the general solutions to this system:

$$\underbrace{\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}}_{x'(t)} = \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x(t)}$$

We can diagonalize A into $A = UDU^{-1}$,

where $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$. $\underbrace{\ }_{v_1}\ \underbrace{\ }_{v_2}\ \underbrace{\ }_{v_3}$.

[$\lambda = 2$, eigenvectors v_1, v_2

$\lambda = 4$, " v_3]

$$x'(t) = Ax(t) = UDU^{-1}x$$

$$\Rightarrow \quad U^{-1}x' = DU^{-1}x.$$

Let $\boxed{y(t) = U^{-1}x(t)}$. Then.

$$y'(t) = D\,y(t)$$

that is, $\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, which is much simpler to solve.

$$\begin{cases} y_1'(t) = 2 y_1(t), \\ y_2'(t) = 2 y_2(t), \\ y_3'(t) = 4 y_3(t). \end{cases} \qquad \begin{aligned} y_1(t) &= c_1 e^{2t}, \\ y_2(t) &= c_2 e^{2t}, \\ y_3(t) &= c_3 e^{4t}, \end{aligned} \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

So,

$$x(t) = V y(t) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{2t} \\ c_3 e^{4t} \end{pmatrix}$$

$$= e^{2t} \underbrace{\left(c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right)}_{\text{in } \ker(A - 2I)} + e^{4t} \left(c_3 \underbrace{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}_{\text{in } \ker(A-4I).} \right)$$

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