Lecture 32: Quick review from previous lecture

- We say that the matrix A is **diagonalizable** if it can be factored in the form $A = VDV^{-1}$ where D is diagonal and V is nonsingular.
- A is complete if every eigenvalue's eigenspace satisfying $\dim V_\lambda =$ the multiplicity of λ. $(\det(A-\lambda I)=0)$ $V_\lambda = \mathrm{Ker}(A-\lambda I)$
- A matrix is complete if and only if it is diagonalizable.
- A^k has the same eigenvectors as A, and the eigenvalues are just $\lambda_1^k, \ldots, \lambda_n^k$.

Today we will discuss diagonalization of symmetric matrices.

$$A = V D V^{-1} = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} [v_1 \cdots v_n]^{-1}.$$

- Lecture will be recorded -

- Solutions for Midterm 2 has been posted on Canvas, see "Announcements".

8.5 Eigenvalues of Symmetric Matrices

Let's focus on the theory of eigenvalues and eigenvectors for **symmetric matrices**, which have many nice properties.

Recall the example again. $A = A^T$.

Example 1. $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. In Lecture 30, we have found

$$\text{eigenvalue } \lambda = 2, \qquad \text{eigenvectors } \mathbf{v}_1 = (-1, 1, 0)^T, \quad \mathbf{v}_2 = (0, 0, 1)^T,$$
$$\text{eigenvalue } \lambda = 4, \qquad \text{eigenvector } \mathbf{v}_3 = (1, 1, 0)^T.$$

Thus, the matrix A is complete. Moreover,

$$A = VDV^{-1}, = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}.$$

where $D = \text{diag}(2, 2, 4)$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

- These eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal!
- The eigenvalues of A are real numbers, not complex.

$v_1 \cdot v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

$v_2 \cdot v_3 = 0$

$v_1 \cdot v_3 = 0.$

These facts are explained by the following Spectral Theorem.

The Spectral Theorem: Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then

entries in A are real number. $A = A^T$

1. All the eigenvalues of A are real.
2. Eigenvectors corresponding to **distinct** eigenvalues are orthogonal.
3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.

In particular, all real symmetric matrices are complete and real diagonalizable.

* Orthogonality is with respect to the standard dot product on $\mathbb{R}^n$.

Its proof will be discussed later.

Suppose A is **real and symmetric**, and let $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues. Then the above Spectral Theorem tells us we can choose eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ (so $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$) that are **orthonormal**.

If $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$A = UDU^{-1},$$

where columns of U are orthonormal basis of $\mathbb{R}^n$.
It implies U is orthogonal matrix, $(U^TU = UU^T = I)$
and then $U^{-1} = U^T$. Thus

$$\boxed{A = UDU^T}.$$

From **Example 1**, we have seen $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has eigenvectors

$$\mathbf{v}_1 = (-1, 1, 0)^T, \quad \mathbf{v}_2 = (0, 0, 1)^T, \quad \mathbf{v}_3 = (1, 1, 0)^T.$$

Normalizing these vectors, we get the matrix

$$Q = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$ is orthogonal matrix

Thus, we have the factorization

$$\begin{aligned} A &= \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \\ &= Q \,\text{diag}(2, 2, 4)\, Q^T \end{aligned}$$

Thus, we conclude that

The Spectral Theorem: Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then there exists an orthogonal matrix Q such that

$$A = QDQ^{-1} = QDQ^T, \qquad \text{(spectral factorization)}$$

where D is a real diagonal matrix. The eigenvalues of A appear on the diagonal of D, while the columns of Q are the corresponding orthonormal eigenvectors.

* The term "spectrum" refers to the eigenvalues of a matrix.

① $K = K^T$
② $x^T K x > 0$ if $x \neq 0$

§ **Revisit Positive definite matrix.** Suppose K is positive definite (in particular, symmetric). Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ denote the orthonormal eigenvector basis, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

$(K > 0)$

Fact 1: A symmetric matrix K is positive definite if and only if all of its eigenvalues are strictly positive, that is, $\lambda_j > 0$

[To see this:]

$(\Longrightarrow)$ Suppose $K > 0$. Thus

$$0 < u_j^T K u_j = u_j^T(\lambda_j u_j) = \lambda_j u_j^T u_j = \lambda_j \|u_j\|^2 = \lambda_j$$

So, $0 < \lambda_j$, $1 \le j \le n$.

$(\Longleftarrow)$ Since $\{u_1, \ldots, u_n\}$ is an orthornormal basis for $\mathbb{R}^n$, for any $x \neq 0$, $x = c_1 u_1 + \cdots + c_n u_n$.

Remark: The same proof shows that K is positive semidefinite if and only if all its eigenvalues $\lambda \geq 0$.

$$x^T K x = (c_1 u_1 + \cdots + c_n u_n)^T K (c_1 u_1 + \cdots + c_n u_n)$$
$$= (c_1 u_1 + \cdots + c_n u_n)^T (c_1 \lambda_1 u_1 + \cdots + c_n \lambda_n u_n)$$
$$= \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 > 0.$$

$\begin{cases} u_i^T u_j = 0, & i \neq j \\ u_j^T u_j = 1. \end{cases}$

Example. In Lecture 30, we have seen that the eigenvalues of $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$ are 2, 4 and 6. Thus, from the ~~Spectral Theorem~~ Fact 1., since all eigenvalues are positive, A is positive definite (or $A > 0$).

*Note that to see if a matrix is positive definite, one can also perform the Gaussian elimination (See In Lecture 19):

From Gaussian elimination. We have

pivots

$$A \xrightarrow{③ + \frac{1}{5} ②} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 24/5 \end{pmatrix}.$$

Since all diagonal entries are positive, we confirm that A is positive definite. □

§ The proof of the Spectral Theorem.

Proof of the spectral theorem. Let $A = A^T$ be a real symmetric $n \times n$ matrix.

1. Show all eigenvalues of A is real.
2. The eigenvectors of A corresponding to different eigenvalues are orthogonal.
3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.

(See Textbook for property 3)

Proof: 1. Since $A = A^T$, one has $\langle Av, w\rangle = \langle v, Aw\rangle$. ①

Suppose λ is an eigenvalue with corresponding eigenvector v.

Recall
$\lambda = a + ib$
$\bar\lambda = a - ib$

$$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle.$$

$$\langle v, Av\rangle = \langle v, \lambda v\rangle = \bar\lambda\langle v, v\rangle$$

□

Thus, $\lambda\langle v, v\rangle = \bar\lambda\langle v, v\rangle$. by ①.

we get $\lambda = \bar{\lambda}$, so λ is real. #

[Proof continue]

2. Suppose $\lambda \neq \mu$. and $Av = \lambda v$, $Aw = \mu w$.

$$\underbrace{\langle \lambda v, w\rangle}_{\lambda\langle v,w\rangle} = \langle Av, w\rangle \underset{A=A^T}{=} \langle v, Aw\rangle = \underbrace{\langle v, \mu w\rangle}_{\mu\langle v,w\rangle \text{ (}\mu\text{ is real.)}}$$

$$\underbrace{(\lambda-\mu)}_{\neq 0}\langle v,w\rangle = 0.$$ Then $\langle v,w\rangle = 0$. #

Fact: If $A = A^T$ is symmetric, suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the orthonormal eigenvectors. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ all have non-zero eigenvalues, but $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n$ have eigenvalue 0 (i.e. they're in ker A). Consequently, $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are orthogonal to ker A, and hence

$$\mathbf{u}_1, \ldots, \mathbf{u}_r \text{ form an orthonormal basis for coimg } A = \text{img } A.$$

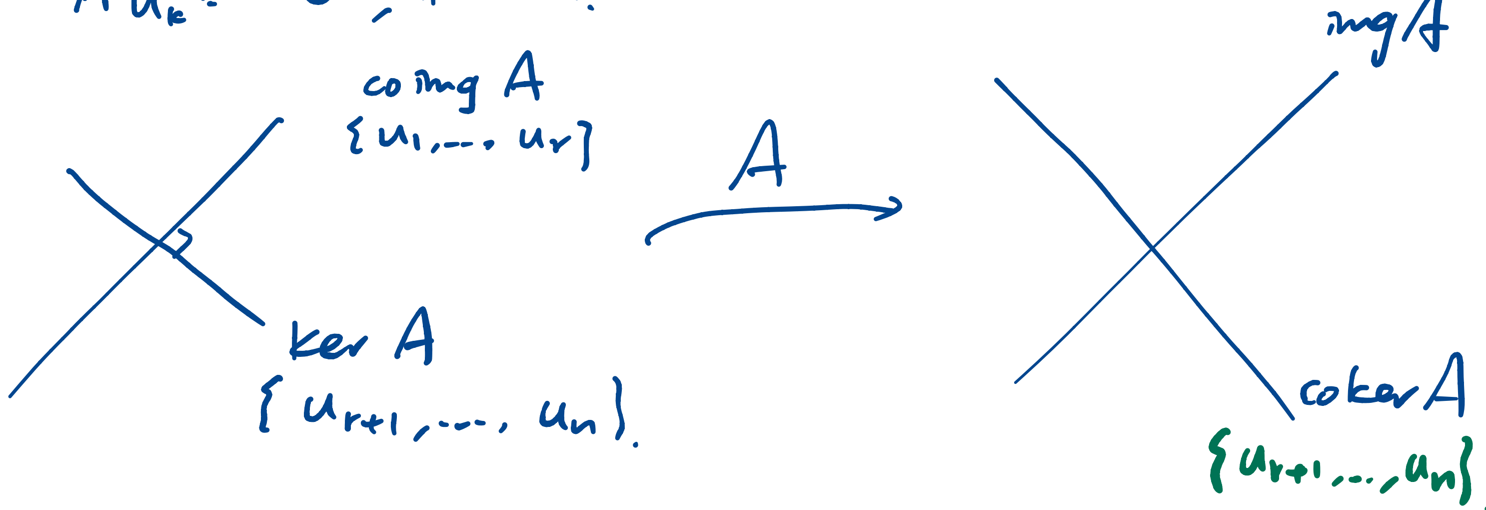
Moreover, one has

$$\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n \text{ form an orthonormal basis for ker } A = \text{coker } A.$$

$A u_j = \overset{\neq 0}{\boxed{\lambda_j}} u_j$, $1 \leq j \leq r$

$A u_k = 0$, $r+1 \leq k \leq n$.

$A v_j = \lambda_j v_j$ span$\{Au_1, \ldots, Au_r\}$ $=$ span$\{u_1, \ldots, u_r\}$



Example. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find an orthonormal basis for coimg A.

$0 = \det(A - \lambda I) = \lambda(\lambda - 2)(\lambda - 3).$

$\lambda = 0,\ 2,\ 3.$

$\underline{\lambda = 0}$. $\operatorname{Ker}(A - 0I) = \operatorname{Ker} A$. eigenvector

$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$

$\underline{\lambda = 2}$, $\operatorname{Ker}(A - 2I) = \operatorname{Ker}\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

eigenvector $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$

$\underline{\lambda = 3}$,

$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

coimg A

$\{v_1, v_2\}$

ker A, $\{v_3\}$

Exercise.

$\underline{\text{Ans}}$: $\left\{ \dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|} \right\}$. #

Warning: Do not confuse the $L\tilde{D}L^T$ factorization(from Gaussian elimination) of regular symmetric matrices with the QDQ^T spectral factorization of a symmetric matrix from the spectral theorem!

Note that: If $A = L\tilde{D}L^T$ (from Gaussian elimination), where L is lower triangular, the diagonal matrix $\tilde{D}$ will typically *not* contain the eigenvalues on its diagonal.