Lecture 33: Quick review from previous lecture

• The Spectral Theorem: Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then

1. All the eigenvalues of $A$ are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of $n$ eigenvectors of $A$.
4. Then there exists an orthogonal matrix $Q$ such that

$$A = QDQ^{-1} = QDQ^T,$$

where $D$ is a real diagonal matrix.

• A symmetric matrix $K$ is positive definite if and only if all of its eigenvalues are strictly positive, that is, $\lambda_j > 0$

Today we will discuss diagonalization of symmetric matrices.

$$A = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} [u_1 \cdots u_n]^T$$

$$= QDQ^T.$$ 

- Lecture will be recorded -

• Instructions and details for Final Exam has been posted on Canvas, see ”Announcements”.
**Warning:** Do not confuse the $L\tilde{D}LT$ factorization (from Gaussian elimination) of regular symmetric matrices with the $QDQT$ spectral factorization of a symmetric matrix from the spectral theorem!

Note that: If $A = L\tilde{D}LT$ (from Gaussian elimination), where $L$ is lower triangular, the diagonal matrix $\tilde{D}$ will typically *not* contain the eigenvalues on its diagonal. See the following example.

**Example.** The $L\tilde{D}LT$ factorization of $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ is $A = LU = L\tilde{D}LT = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \end{pmatrix}$.

However, the spectral factorization of $A$ is $A = \det(A - \lambda I) = (\lambda - 6)(\lambda - 2)$, $\lambda = 6, 2$.

**Some Geometric observation.**

In $\mathbb{R}^2$, the spectral factorization of a symmetric matrix $A$ has a natural geometric interpretation.

Suppose that $A$ is an symmetric $2 \times 2$ matrix. Let $\mathbf{u}, \mathbf{v}$ be the orthonormal eigenvector basis in $\mathbb{R}^2$ with eigenvalues $\lambda, \mu$.

**Q:** What is the image of the matrix $A$ on the unit circle in $\mathbb{R}^2$?
Thus, we have shown that $A$ maps the unit circle in $\mathbb{R}^2$ to the ellipse

$$\{ a u + b v : \frac{a^2}{\lambda^2} + \frac{b^2}{\mu^2} = 1 \}$$

The principal directions of this ellipse are the eigenvectors $u$ and $v$, and the principal stretches are the eigenvalues $\lambda$ and $\mu$.

So the eigenvectors and values provide a geometric interpretation of the action of $A$ on $\mathbb{R}^2$.

**Example.** Consider the quadratic form

$$q(x) = \langle Ax, x \rangle = x^T A x.$$ 

Using the spectral factorization to diagonalize this quadratic form.

$A$ is real symmetric matrix, $O = \det(A - \lambda I) \Rightarrow \lambda = 2, 4$

$\lambda = 2 : \quad A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad u_1 = (1, -1) \Rightarrow u_1 = \frac{1}{\sqrt{2}} (1, -1)$.

$\lambda = 4 : \quad A - 4I = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad u_2 = (1, 1) \Rightarrow u_2 = \frac{1}{\sqrt{2}} (1, 1)$.

So, spectral factorization of $A$ is

$$A = Q \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} Q^T, \quad \text{where} \quad Q = [u_1, u_2].$$

$$q(x) = x^T A x = x^T Q D Q^T x = (Q^T x)^T D (Q^T x)$$

Let $y = Q^T x$. Then $q(x) = y^T D y = 2 y_1^2 + 4 y_2^2$. 

$y^T = Q^T x$ is coordinate of $x$ to orthonormal basis $\{u_1, u_2\}$.
§ Revisit Matrix norm.
The eigenvalues of a symmetric matrix $A$ allow us to compute its matrix norm and its Frobenius norm easily.

§ Frobenius norm.
Recall that the Frobenius norm of a matrix $A$ is defined by

$$
\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2}
$$

Then

Let $A$ be a real, symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$
\|A\|_F = \sqrt{\sum_{i=1}^{n} \lambda_i^2}
$$

[To see this] Let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$, we can factorize $A$ into $A = QDQ^T$, $Q$ : orthogonal, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Since $Q$ is orthogonal, $\|QA\|_F = \|AQ\|_F = \|A\|_F$.

\[\text{proof:} \]

\[\|QA\|_F^2 = \|[Qa_1, \ldots, Qa_n]\|_F^2 = \|Qa_1\|_2^2 + \cdots + \|Qa_n\|_2^2 = \|a_1\|_2^2 + \cdots + \|a_n\|_2^2 = \|A\|_F^2. \]

Now

$$
\|A\|_F = \|QDQ^T\|_F = \|DQ^T\|_F = \|D\|_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}
$$
Example. We found the eigenvalues of \( A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \) to be 0, 2 and 3.

The sum of squares of the eigenvalues is \( 2^2 + 3^2 = 13 \).

The squared Frobenius norm of the entries of \( A \) is

\[
\| A \|_F^2 = 1^2 + (-1)^2 + (-1)^2 + 1^2 + 3^2 = 13, \quad = 2^2 + 3^2
\]

verifying the general formula in this case.

§ Natural Matrix norm. More interesting is using the eigenvalues to compute the matrix norm of a real symmetric matrix \( A \):

Let’s consider \( \| x \|_2 = \sqrt{x_1^2 + \ldots + x_n^2} \). The natural matrix norm of \( A \) is

\[
\| A \| = \max \{ \| Au \|_2 : \| u \|_2 = 1 \}.
\]

We write \( A = QDQ^T \), \( Q \) : orthogonal, \( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} \).

\[
\| u \|_2 = 1, \quad \| Au \|_2 = \|QDQ^T u \|_2 = \| DQ^T u \|_2 = \| D \tilde{u} \|_2, \text{ where}
\]

\[
\tilde{u} = Q^T u. \quad \text{Then} \quad \| \tilde{u} \|_2 = \| Q^T u \|_2 = \| u \|_2 = 1.
\]

Thus \( \| A \| = \max \{ \| Au \|_2 : \| u \|_2 = 1 \} \).

\[
= \max \{ \| D \tilde{u} \|_2 : \| \tilde{u} \|_2 = 1 \} = \| D \| = \max_{1 \leq i \leq n} | \lambda_i |.
\]

Therefore, we conclude that

\[
\| A \| = \max_{1 \leq i \leq n} | \lambda_i |
\]
**Example.** We found the eigenvalues of \( A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \) to be 0, 2 and 3. Then the natural matrix norm of \( A \) is 3.

\[
\| A \| = \max \{ |101|, |121|, |31| \} = 3
\]

**Q:** Which vector in \( \mathbb{R}^n \) undergoes the largest change in length under the action of \( A \)? That is, which unit vector \( x \) maximizes \( \| Ax \|_2 \)?

**Fact:** Suppose that a symmetric matrix \( A \) has real eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfying

\[
|\lambda_1| \geq \cdots \geq |\lambda_n|.
\]

with corresponding orthonormal eigenvectors \( u_1, \ldots, u_n \). Then

\[
|\lambda_1| = \max \{ \| Ax \|_2 : \| x \|_2 = 1 \}, \quad |\lambda_n| = \min \{ \| Ax \|_2 : \| x \|_2 = 1 \}. \tag{1.}
\]

The **maximal value** of \( \| Ax \|_2 \) is achieved when \( x = \pm u_1 \), the unit eigenvector associated with the largest eigenvalue \( \lambda_1 \).

The **minimal value** of \( \| Ax \|_2 \) is achieved when \( x = \pm u_n \), the unit eigenvector associated with the smallest eigenvalue \( \lambda_n \).

**proof:** For any \( x \), \( \| x \|_2 = 1 \), we can write

\[
x = \langle x, u_1 \rangle u_1 + \cdots + \langle x, u_n \rangle u_n. \text{ with } \langle x, u_i \rangle^2 + \cdots + \langle x, u_n \rangle^2 = 1.
\]

1. \( \| Ax \|_2^2 = \| A(\langle x, u_1 \rangle u_1 + \cdots + \langle x, u_n \rangle u_n) \|_2^2 = \| \langle x, u_1 \rangle \lambda_1 u_1 + \cdots + \langle x, u_n \rangle \lambda_n u_n \|_2^2 \leq \lambda_1^2 \langle x, u_1 \rangle^2 + \cdots + \lambda_n^2 \langle x, u_n \rangle^2 \leq \lambda_1^2 \langle x, u_1 \rangle^2 + \lambda_1 \langle x, u_2 \rangle^2 + \cdots + \lambda_n^2 \langle x, u_n \rangle^2 = \lambda_1^2 \langle x, u_1 \rangle^2 + \cdots + \langle x, u_n \rangle^2 \leq \lambda_1^2 \langle x, u_1 \rangle^2 + \cdots + \langle x, u_n \rangle^2 = \lambda_1^2 \langle x, u_1 \rangle^2 + \cdots + \langle x, u_n \rangle^2 = \lambda_1^2. \]

So, \( \max \{ \| Ax \|_2 : \| x \|_2 = 1 \} \leq |\lambda_1| \). \( \| Au_1 \| = |\lambda_1| \| u_1 \| = |\lambda_1| \).

Thus, \( \max \{ \| Ax \|_2 : \| x \|_2 = 1 \} = |\lambda_1| \).

2. Similarly, we can show (2).