Lecture 33: Quick review from previous lecture

- **The Spectral Theorem:** Let $A = A^T$ be a real symmetric $n \times n$ matrix. Then
  1. All the eigenvalues of A are real.
  2. Eigenvectors corresponding to **distinct** eigenvalues are orthogonal.
  3. There is an orthonormal basis of $\mathbb{R}^n$ consisting of n eigenvectors of A.
  4. Then there exists an orthogonal matrix Q such that

     $$A = QDQ^{-1} = QDQ^T, \qquad \underline{\text{(spectral factorization)}}$$

     where D is a real diagonal matrix.
- A symmetric matrix K is positive definite if and only if all of its eigenvalues are strictly positive, that is, $\lambda_j > 0$

Today we will discuss diagonalization of symmetric matrices.

$$A = \overset{Q}{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \overset{Q^{-1}}{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^{-1}}$$

$$= Q D Q^T.$$

- Lecture will be recorded -

- Instructions and details for Final Exam has been posted on Canvas, see "Announcements".

Warning: Do not confuse the $L\tilde{D}L^T$ factorization(from Gaussian elimination) of regular symmetric matrices with the QDQ^T spectral factorization of a symmetric matrix from the spectral theorem!

Note that: If $A = L\tilde{D}L^T$ (from Gaussian elimination), where L is lower triangular, the diagonal matrix $\tilde{D}$ will typically *not* contain the eigenvalues on its diagonal.

See the following example.

Example. The $L\tilde{D}L^T$ factorization of $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ is

$A \xrightarrow{②-\frac{1}{2}①} \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} = U$

$A = LU = L\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} L^T$

$L = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

pivots

However, the spectral factorization of A is

eigenvalues

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$0 = \det(A - \lambda I) = (\lambda - 6)(\lambda - 2)$, $\lambda = 6, 2$.

§ Some Geometric observation.

In $\mathbb{R}^2$, the spectral factorization of a symmetric matrix A has a natural geometric interpretation.

Suppose that A is an symmetric 2×2 matrix. Let $\mathbf{u}, \mathbf{v}$ be the orthonormal eigenvector basis in $\mathbb{R}^2$ with eigenvalues λ, μ.

Q: What is the image of the matrix A on the unit circle in $\mathbb{R}^2$?

Let $\vec{x} = (x, y)$ in $\mathbb{R}^2$ with $x^2 + y^2 = 1$. We can write

$\vec{x} = \underline{\langle \vec{x}, u \rangle u + \langle \vec{x}, v \rangle v}$. $\|\vec{x}\|^2 = \langle \vec{x}, u \rangle^2 + \langle \vec{x}, v \rangle^2 = x^2 + y^2 = 1$.

$A\vec{x} = A(\langle \vec{x}, u \rangle u + \langle \vec{x}, v \rangle v) = \langle \vec{x}, u \rangle Au + \langle \vec{x}, v \rangle Av$

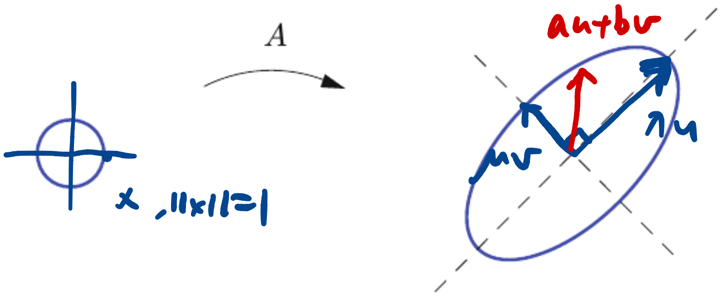
$= \underbrace{\lambda \langle \vec{x}, u \rangle}_{a} u + \underbrace{\mu \langle \vec{x}, v \rangle}_{b} v$.

Thus $\left(\frac{a}{\lambda}\right)^2 + \left(\frac{b}{\mu}\right)^2 = \langle \vec{x}, u \rangle^2 + \langle \vec{x}, u \rangle^2 = 1$

Thus, we have shown that A maps the unit circle in $\mathbb{R}^2$ to the ellipse

$$\left\{ a\mathbf{u} + b\mathbf{v} : \frac{a^2}{\lambda^2} + \frac{b^2}{\mu^2} = 1 \right\}$$

The principal directions of this ellipse are the eigenvectors $\mathbf{u}$ and $\mathbf{v}$, and the principal stretches are the eigenvalues λ and μ.



So the eigenvectors and values provide a geometric interpretation of the action of A on $\mathbb{R}^2$.

Example. Consider the quadratic form $q(x) = \langle Ax, x\rangle = x^T A x.$

$$q(x) = 3x_1^2 + \boxed{2x_1x_2} + \boxed{3x_2^2} = (x_1, x_2)\underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

Using the spectral factorization to diagonalized this quadratic form.

A is real - symmetric matrix. $0 = \det(A - \lambda I) \Rightarrow \lambda = 2, 4.$

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\underline{\lambda = 4}$: $A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

So. spectral factorization of A is

$$A = Q \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} Q^T, \text{ where } Q = [u_1, u_2]. \quad \#$$
$$= Q \; D \; Q^T.$$

$q(x) = x^T A x = x^T QDQ^T x = (Q^T x)^T D (Qx)$

Let $y = Q^T x$. Then $q(x) = y^T D y = 2y_1^2 + 4y_2^2$. #

- $y = Q^T x$ is coordinate of x to orthonormal basis $\{u_1, u_2\}$.
- $x \neq 0$, $q(x) > 0$.

§ Revisit Matrix norm.

The eigenvalues of a symmetric matrix A allow us to compute its matrix norm and its Frobenius norm easily.

$$\begin{array}{cc} a_1 & a_2 \end{array}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\|A\|_F = \sqrt{1^2+2^2+3^2+4^2}.$$

$$= \sqrt{\|a_1\|_2^2 + \|a_2\|_2^2}$$

- $\|a_1\|_2^2 = 1^2+3^2$

  $\|a_2\|_2^2 = 2^2+4^2.$

§ Frobenius norm.

Recall that the **Frobenius norm** of a matrix A is defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}$$

Then

> Let A be a real, symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then
>
> $$\|A\|_F = \sqrt{\sum_{i=1}^{n} \lambda_i^2}$$

[To see this] let $A = [a_1 \cdots a_n]$. we can factorize A into $A = QDQ^T$, Q: orthogonal, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Since Q is orthogonal, $\underline{\|QA\|_F = \|AQ\|_F = \|A\|_F}$.

proof:

$$\|QA\|_F^2 = \|[Qa_1, \cdots, Qa_n]\|_F^2 = \|Qa_1\|_2^2 + \cdots + \|Qa_n\|_2^2$$
$$= \|a_1\|_2^2 + \cdots + \|a_n\|_2^2$$
$$= \|A\|_F^2.$$

Now

$$\|A\|_F = \|\underline{Q}D\underline{Q}^T\|_F = \|DQ^T\|_F = \|D\|_F$$
$$= \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$$

※.

Example. We found the eigenvalues of $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ to be 0, 2 and 3.

The sum of squares of the eigenvalues is $\underline{2^2 + 3^2 = 13.}$
The squared Frobenius norm of the entries of A is

$$\|A\|_F^2 = \underline{1}^2 + \underline{(-1)}^2 + \underline{(-1)}^2 + \underline{1}^2 + \underline{3}^2 = \underline{13}, = 2^2 + 3^2$$

verifying the general formula in this case.

§ **Natural Matrix norm.** More interesting is using the eigenvalues to compute the matrix norm of a real symmetric matrix A:
Let's consider $\|\mathbf{x}\|_2 = \underline{\sqrt{x_1^2 + \ldots + x_n^2}}$. The natural matrix norm of A is

$$\|A\| = \max\{\underline{\|A\mathbf{u}\|_2} : \underline{\|\mathbf{u}\|_2 = 1}\}.$$

We write $A = QDQ^T$, Q: orthogonal, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

$\|u\|_2 = 1$. $\|Au\|_2 = \|QDQ^Tu\|_2 = \|DQ^Tu\|_2$.
$= \|D\tilde{u}\|_2$, where

$\tilde{u} = Q^Tu$. Then $\|\tilde{u}\|_2 = \|Q^Tu\|_2 = \|u\|_2 = 1$.

Thus, $\|A\| = \max\{\|Au\|_2 \mid \|u\|_2 = 1\}$.
$= \max\{\|D\tilde{u}\|_2 \mid \|\tilde{u}\|_2 = 1\} = \|D\| = \max_{1 \le j \le n} |\lambda_j|$.

Therefore, we conclude that

$$\boxed{\|A\| = \max_{1 \le i \le n} |\lambda_i|}$$

Example. We found the eigenvalues of $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ to be 0, 2 and 3.

Then the natural matrix norm of A is 3.

$$\|A\| = \max\{|0|, |2|, |3|\} = \underline{3} \ \#.$$

Q: Which vector in $\mathbb{R}^n$ undergoes the largest change in length under the action of A? That is, which unit vector $\mathbf{x}$ maximizes $\|A\mathbf{x}\|_2$?

Fact: Suppose that a symmetric matrix A has real eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$

with corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Then

$$|\lambda_1| = \max\{\underline{\|A\mathbf{x}\|_2} : \ \|\mathbf{x}\|_2 = 1\}, \qquad |\lambda_n| = \min\{\|A\mathbf{x}\|_2 : \ \|\mathbf{x}\|_2 = 1\}.$$

(1.) The **maximal value** of $\|A\mathbf{x}\|_2$ is achieved when $\mathbf{x} = \pm\mathbf{u}_1$, the unit eigenvector associated with the ~~largest~~ eigenvalue λ_1.

(2.) The **minimal value** of $\|A\mathbf{x}\|_2$ is achieved when $\mathbf{x} = \pm\mathbf{u}_n$, the unit eigenvector associated with the ~~smallest~~ eigenvalue λ_n.

proof: For any x. $\|x\|_2 = 1$, we can write

$x = \langle x, u_1\rangle u_1 + \cdots + \langle x, u_n\rangle u_n$. with $\langle x,u_1\rangle^2 + \cdots + \langle x,u_n\rangle^2 = 1$.

1. $$\|Ax\|_2^2 = \|A(\langle x, u_1\rangle u_1 + \cdots + \langle x, u_n\rangle u_n)\|_2^2$$

$$= \|\langle x,u_1\rangle \lambda_1 \underline{u_1} + \cdots + \langle x, u_n\rangle \lambda_n \underline{u_n}\|_2^2$$

$$= \lambda_1^2 \langle x,u_1\rangle^2 + \cdots + \lambda_n^2 \langle x,u_n\rangle^2. \qquad 2 \text{ since } |\lambda_j| \leq |\lambda_1|.$$

$$\leq \lambda_1^2 \langle x,u_1\rangle^2 + \lambda_1 \langle x,u_2\rangle^2 + \cdots + \lambda_1^2 \langle x,u_n\rangle^2$$

$$= \lambda_1^2(\langle x,u_1\rangle^2 + \cdots + \langle x,u_n\rangle^2) = \lambda_1^2.$$

So. $\max\{\|Ax\|_2 \mid \|x\|_2 = 1\} \leq |\lambda_1|$. $\|Au_1\| = |\lambda_1| \|u_1\| = |\lambda_1|$.

Thus. $\max\{\|Ax\|_2 \mid \|x\|_2 = 1\} = |\lambda_1|$. #.

2. Similarly. we can show (2) #