# Lecture 38: Quick review from previous lecture

- The **condition number** of a nonsingular $n \times n$ matrix is the ratio between its largest and smallest singular values, namely,

$$\kappa(A) = \frac{\sigma_1}{\sigma_n}.$$

- If $A$ is $n \times n$ nonsingular matrix, then $A$ and $A^{-1}$ have the same condition number.
- If $A$ is $n \times n$ nonsingular matrix, then

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2.$$

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Today we will review some concepts.

**- Lecture will be recorded -**

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- Information for Final Exam and Course Grade has been posted on Canvas, see "Announcements".

**Problem 1:** Find the permuted $LU$ factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Clearly identify the permutation matrix $P$, the lower unitriangular matrix $L$, and the upper triangular matrix $U$.

$$A \xrightarrow[③-2①]{②-①} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{②\leftrightarrow③} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[③+2①]{②+①} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = L$$

$PA = LU$, where $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

**Problem 2:** Find $A^{-1}$, where $A$ is the same matrix from Problem 1.

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[③-2①]{②-①} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{②\leftrightarrow③} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-②} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{①-②} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right).$$

$$A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \#$$

**Problem 3:** Solve the system $A\mathbf{x} = \mathbf{b}$, where $A$ is the same matrix from Problem 1, and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$$x = A^{-1} b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

**Problem 4:** Find all solutions to the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

$\vec{x} = (x, y, z, w)$

Augmented matrix

$$(A \mid b) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & 2 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{③ - ①} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right)$$

3rd equation: $2z = 2 \Rightarrow z = 1$

2nd " : $y = 1 + w$

1st " : $x = 1 - y - w = -2w$

Then solutions are

$$\vec{x} = \begin{pmatrix} -2w \\ 1+w \\ 1 \\ w \end{pmatrix}, \quad w \in \mathbb{R}$$

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**Problem 5:** Suppose $A$ is an $n$-by-$n$ matrix satisfying $A^4 - A^2 + 2A + I = O$. Find an expression for $A^{-1}$ in terms of $A$. $(AX = XA = I_n,\ X = A^{-1})$

$$A^4 - A^2 + 2A + I = 0$$

$$I = -A^4 + A^2 - 2A.$$

$$= A(\underbrace{-A^3 + A - 2I}_{A^{-1}}).$$ Then $A^{-1} = -A^3 + A - 2I$.

**Problem 6:** Suppose $A$ and $B$ are 2-by-2 matrices whose product is $A(5B)^3 = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$. If $\det(A) = -2$, what is $\det(B)$?

$$2 = \det(A(5B)^3) = \det A \det(5B)^3 = \overbrace{\det A}^{-2}\ [\det(5B)]^3$$

$$[\det(5B)]^3 = -1. \quad \underset{5^2 \det B}{\underset{\parallel}{\det(5B)}} = -1.$$

$$\Rightarrow \det B = -1/25.\#$$

**Problem 7:** Using the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$, compute the orthogonal projection of $p(t) = t^2$ onto $\text{span}\{1, t^3\}$.

Orthogonal projection is

$$\frac{\langle p(t), 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle p(t), t^3\rangle}{\langle t^3, t^3\rangle} t^3 = \frac{\frac{2}{3}}{2} = \frac{1}{3}.\#$$

- $\langle p(t), 1\rangle = \int_{-1}^{1} t^2 \cdot 1\, dt = \frac{1}{3}t^3\Big|_{-1}^{1} = \frac{2}{3}.$
- $\langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = t\Big|_{-1}^{1} = 2.$
- $\langle p(t), t^3\rangle = \int_{-1}^{1} t^5\, dt = \frac{1}{6}t^6\Big|_{-1}^{1} = 0.$

**Problem 8:** Find the symmetric 3-by-3 matrix $K$ satisfying

$$\mathbf{x}^T K\mathbf{x} = 2x_1^2 + 4x_1x_2 + 3x_2^2 - 2x_1x_3 + \frac{3}{2}x_3^2 - 2x_2x_3$$

for all vectors $\mathbf{x} = (x_1, x_2, x_3)^T$. Show $K$ is positive definite and write $K = \cancel{B^2}\ BB^T$.

1) $K = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & \frac{3}{2} \end{bmatrix}$

2) $K \xrightarrow[③+\frac{1}{2}①]{②-①} \begin{pmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U$. All positive pivots, so $K > 0$.

3) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$. $K = LU = L\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$

$= \underbrace{L\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{B}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L^T$

$= B\ B^T$

**Problem 9:**

**a)** Find the dimension of the subspace $V$ of all vectors $(x, y, z, w)^T$ in $\mathbb{R}^4$ satisfying $2x - 4y + 3z + w = 0$, $3x - 5y + z = 0$, and $5x - 9y + 4z + w = 0$.

**b)** Find the dimension of $V^\perp$.

a) $\underbrace{\begin{pmatrix} 2 & -4 & 3 & 1 \\ 3 & -5 & 1 & 0 \\ 5 & -9 & 4 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$A \longrightarrow \begin{pmatrix} 2 & -4 & 3 & 1 \\ 3 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So $\dim V = 2$ #

b) $\dim V^\perp = 4 - 2 = 2$. #

**Problem 10:** Compute the rank of the matrix $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ 3 & 2 & -4 \end{pmatrix}$.

$$A \longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{rank } A = 2.$$

**Problem 11:** Find the dimensions of each of the four fundamental subspaces of the matrix $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$.

$$A \longrightarrow \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{rank } A = 2.$$

| | dim |
|---|---|
| img $A$ | 2 |
| coimg $A$ | 2 |
| ker $A$ | $4-2=2$. |
| coker $A$ | $3-2=1$. |

$$\mathbb{R}^4 \xrightarrow{A} \mathbb{R}^3$$

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**Problem 12:** Find an orthonormal basis for the cokernel of $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -2 & -4 & 2 \\ -1 & -2 & 1 \end{pmatrix}$

$\operatorname{coker} A = \ker A^T$.

$$A^T = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 2 & 1 & -4 & -2 \\ -1 & -1 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\ker A^T$ has points $\begin{pmatrix} 2z+w \\ 0 \\ z \\ w \end{pmatrix} = z\underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + w\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$.

$\{v_1, v_2\}$ is basis for $\ker A^T$.

By Gram-Schmidt process,

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix}. \#$$

**Problem 13:** If $A = A_{n\times n}$ is positive semidefinite, then $B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}_{(n+1)\times(n+1)}$ is positive semidefinite.

① $B^T = \begin{pmatrix} 1 & 0 \\ 0 & A^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = B$, so $B$ is symmetric

② Taking $x \in \mathbb{R}^n$, $y \in \mathbb{R}$.

$A \geq 0$, so $x^T A x \geq 0$.

$$\begin{pmatrix} y \\ x \end{pmatrix}^T B \begin{pmatrix} y \\ x \end{pmatrix} = (y \;\; x^T) \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}$$

$$= \underset{\geq 0}{y^2} + \underbrace{x^T A x}_{\geq 0} \geq 0.$$

So, $B \geq 0$. #