Lecture 39: Quick review from previous lecture

- We have reviewed permuted LU factorization, inverse of a matrix, positive (semi)definite, determinant, solving a linear system and many others.

Today we will review some concepts.

- Lecture will be recorded -

- Information for Final Exam and Course Grade has been posted on Canvas, see "Announcements".

Problem 14: Find the QR factorization of $A = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -3 \\ 2 & -1 & -3 \end{pmatrix}.$$

Clearly identify the orthogonal matrix Q and the upper triangular matrix R.

$$v_1 = a_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \underline{\underline{q_1}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = a_2 - \frac{\langle a_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\underline{q_2}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = a_3 - \frac{\langle a_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle a_3, v_2 \rangle}{\|v_2\|} v_2 = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \quad \underline{\underline{q_3}} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Then
$$Q = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}. \qquad r_{ii} = \|v_i\|, \quad r_{ij} = \langle a_j, q_i \rangle$$

$$= \begin{pmatrix} 2 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \quad \#$$

Problem 15:

a) Find the matrix norm of $A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, with respect to the standard Euclidean norm $\|\mathbf{y}\|_2 = \sqrt{y_1^2 + y_2^2 + y_3^2}$ on $\mathbb{R}^3$.

$$\|A\| = \max_{1 \le i \le n} |\lambda_i| = 7.$$

$\langle x, x \rangle = \|x\|^2$.

b) Suppose $\|\mathbf{x}\| = 3$ and $\|\mathbf{y}\| = 1$. What is the maximum possible value for $\langle \mathbf{x}, \mathbf{y} \rangle$? What relationship must hold between $\mathbf{x}$ and $\mathbf{y}$ if this value is to be achieved?

1) $|\langle x, y \rangle| \leq \|x\| \|y\| = 3$. maximum value is 3. #

2) $\underline{x = 3y}$. #

c) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, $\|\mathbf{x}\| = 2$, $\|\mathbf{y}\| = 1$, and $\mathbf{z} = 2\mathbf{x} - 3\mathbf{y}$. What is $\langle \mathbf{x}, \mathbf{z} \rangle$? What is $\langle \mathbf{y}, \mathbf{z} \rangle$? What is $\|\mathbf{z}\|$?

1) $\langle x, z \rangle = \langle x, 2x - 3y \rangle = \langle x, 2x \rangle - \langle x, 3y \rangle = 2\langle x, x \rangle - 3\cancel{\langle x, y \rangle}^{0}$
$= 2\|x\|^2 = \underline{8}$ #

2) $\langle y, z \rangle = -3$.

3) $\|z\|^2 = \langle z, z \rangle = \langle 2x - 3y, 2x - 3y \rangle = 4\langle x, x \rangle - \cancel{6\langle x, y \rangle}^{0}$
$- \cancel{6\langle y, x \rangle}^{0} + 9\langle y, y \rangle$
$= 4 \cdot 2^2 + 9 = 25$.

Problem 16: Find all vectors in $\mathbb{R}^3$ orthogonal to both $(1, 2, 0)^T$ and $(2, 5, 2)^T$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \qquad\qquad\quad \|$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad v_1 \qquad\qquad v_2$.

$v_1^T \vec{x} = 0$

$v_2^T \vec{x} = 0$

$. \quad \overbrace{\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \end{pmatrix}}^{A} \vec{x} = 0.$

$A \xrightarrow{②-2①} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}. \quad \vec{x} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} z, \quad z \in \mathbb{R}.$

Problem 17: Write down the 2-by-2 matrix A satisfying $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = \mathbf{w}_2$, where $\mathbf{v}_1 = (1,1)^T$, $\mathbf{v}_2 = (-1,1)^T$, $\mathbf{w}_1 = (1,1)^T$, and $\mathbf{w}_2 = (-2,-2)^T$.

$Av_1 = w_1$

$Av_2 = w_2$: $A\underbrace{[v_1 \ \ v_2]}_{S} = [w_1 \ \ w_2]$

$$A = [w_1\ w_2]\, S^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}. \ \#$$

Problem18: Find a 2-by-2 matrix A with eigenvalues 2 and -3 and corresponding eigenvectors $\underset{v_1}{\underset{\parallel}{(1,-1)^T}}$ and $\underset{v_2}{\underset{\parallel}{(1,0)^T}}$.

$Av_1 = 2v_1$

$Av_2 = -3v_2$: $A\underbrace{[v_1 \ \ v_2]}_{S} = [v_1 \ \ v_2]\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

$$A = S\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} S^{-1} = \begin{pmatrix} -3 & -5 \\ 0 & 2 \end{pmatrix}. \ \#$$

Problem 19: Write out the SVD of the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

$A^TA = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. $\quad 0 = \det(A^TA - \lambda I)$

$\lambda = \underline{\underline{4}}, \ 0.$

$\underline{\lambda = 4}$. $(A^TA - 4I)v = 0$, $v = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$u = \frac{Av}{\sqrt{4}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boxed{\sigma = 2}$

Reduced SVD of A is $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}[2]\left(\frac{1}{\sqrt{2}} \ \frac{-1}{\sqrt{2}}\right)$. #

Problem 20: Find a 2-by-3 matrix A having rank 1 whose singular value is 2, left singular vector is $\mathbf{u} = (1,2)^T/\sqrt{5}$, and right singular vector is $\mathbf{v} = (1,0,1)^T/\sqrt{2}$, that is, $A\mathbf{v} = 2\mathbf{u}$.

$$A = \underline{2uv^T} = \frac{2}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$= u[2]v^T$ (Reduced SVD)

$$= \frac{2}{\sqrt{10}}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 0 & 1\end{pmatrix}$$

$$= \frac{2}{\sqrt{10}}\begin{pmatrix}1 & 0 & 1\\ 2 & 0 & 2\end{pmatrix}. \quad \#$$

Problem 21: Suppose A is a 2-by-2 real matrix for which $1-2i$ is an eigenvalue. Find the trace and determinant of A.

$1 \pm 2i$ are eigenvalues of A.

$$\det A = \lambda_1 \cdots \lambda_n = (1+2i)(1-2i) = \underline{5} \quad \#$$

$$\text{Trace of } A = a_{11} + \dots + a_{nn} = \left(a_{ij}\right)_{n\times n}.$$

$$= \lambda_1 + \lambda_2$$

$$= (1+2i) + (1-2i) = \underline{2}. \quad \#$$

Problem 22: Suppose A is a 2-by-2 symmetric matrix with eigenvalues 3 and -4. Find the operator norm of A and the Frobenius norm of A.

Operator norm of A: $\|A\| = \max_{1\le i\le n} |\lambda_i| = \underline{4}$

Frobenius norm of A: $\|A\|_F = \sqrt{\lambda_1^2 + \dots + \lambda_n^2}$

$$= \underline{5} \quad \#.$$

Problem 23: Suppose A has characteristic polynomial $p_A(\lambda) = \lambda^2 - 2\lambda + \boxed{2}$. Find the determinant of A.

$$\det A = \lambda_1 \lambda_2 = 2 . \#$$

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \cdots + \boxed{\lambda_1 \lambda_2}$$

Problem 24: Suppose A has characteristic polynomial $\underbrace{p_A(\lambda)}_{= \det(A - \lambda I)} = \lambda^2 - 2\lambda + 2$. Find the characteristic polynomial of A^{-1}.

$$p_{A^{-1}}(\tilde\lambda) = \det(A^{-1} - \tilde\lambda I) = \det(A^{-1}(I - \tilde\lambda A))$$

$$= \det A^{-1} \cdot \det(I - \tilde\lambda A)$$

$$= \det(A^{-1}) \cdot \det\left((-\tilde\lambda)\left(A - \frac{1}{\tilde\lambda} I\right)\right)$$

$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{2}$

$$= \det(A^{-1})(-\tilde\lambda)^2 \det\left(A - \frac{1}{\tilde\lambda} I\right) \to p_A\left(\frac{1}{\tilde\lambda}\right)$$

$$= \frac{1}{2} \quad \tilde\lambda^2 \quad \left(\left(\frac{1}{\tilde\lambda}\right)^2 - \frac{2}{\tilde\lambda} + 2\right) = \tilde\lambda^2 - \tilde\lambda + \frac{1}{2} . \#$$

Problem 25: Suppose $A = A^T$ is a symmetric 2-by-2 matrix, and $\det A = 6$. Suppose that $A\mathbf{v} = 2\mathbf{v}$, where $\mathbf{v} = (1,1)^T$. Write an spectral factorization of A.

$$6 = \det A = 2 \cdot \lambda \Rightarrow \underline{\lambda = 3}.$$

v_2: Eigenvector of $\lambda = 3$. v_2 is orthogonal to $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Taking $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^{-1} . \#$$

Problem 26: Let $V = \mathcal{P}^{(1)}$ be the space of polynomials of degree ≤ 1, and $W = \mathcal{P}^{(2)}$ be the space of polynomials of degree ≤ 2. Let $L[p](x) = \int_0^x p(t)dt$ denote the integration operator. Find the matrix representation of L in the monomial bases of V and W.

$P^{(1)} = \{x, 1\}$, $P^{(2)} = \{x^2, x, 1\}$

$L[x] = \int_0^x t\,dt = \frac{1}{2}x^2 + 0\cdot x + 0\cdot 1$

$L[1] = \int_0^x 1\,dt = 0\,x^2 + x + 0\cdot 1$

Thus, matrix representation of L is

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad \#$$

Problem 27: Suppose A is a 3-by-3 matrix with singular values 1,2, and 3. What is the condition number of A? What are the singular values of A^{-1}? What are the singular values of A^T? What is the determinant of A?

s.u

1) $K(A) = 3/1 = 3$

2) s.v. of $A^{-1} = 1, \frac{1}{2}, \frac{1}{3}$

3) s.v. of $A^T = 1, 2, 3.$

4) $\det A = \det(U\Sigma V^T) = \pm 6$ (full SVD) $\#$

[NOTE that $U^TU = UU^T = I$

$1 = \det U^T \det U = (\det U)^2$

$\det U = \pm 1.$]

Similarly, $\det V = \pm 1.$

Euclidean matrix norm

Problem 28: Suppose A is a matrix with singular values 2, 3 and 8. Suppose $\mathbf{u}$ and $\mathbf{v}$ are the left and right singular vectors of A with singular value 8, and let $B = 8\mathbf{u}\mathbf{v}^T$. Find $\|A - B\|_2$ and $\|A - B\|_F$.

$$A = 8uv^T + 3u_2v_2^T + 2u_3v_3^T.$$

$$A - B = 3u_2v_2^T + 2u_3v_3^T.$$

So, $\|A-B\|_2 = 3$

$$\|A-B\|_F = \sqrt{2^2+3^2} = \sqrt{13}.$$

Problem 29: Suppose $A = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u} = (1,-1)^T/\sqrt{2}$ and $\mathbf{v} = (1,1)^T/\sqrt{2}$. Let $\mathbf{b} = (1,0)^T$. Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$. That is, find all vectors $\mathbf{x}$ that minimize $\|A\mathbf{x} - \mathbf{b}\|_2$. Also, find the unique vector $\mathbf{x}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|_2$ and has the smallest Euclidean norm.

1) $A^+ = v_{2\times 1}\,[1]\,u^T_{1\times 2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$x^* = A^+ b = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, has the smallest Euclidean norm.

2) All least squares solution are $x^* + z$, where $z \in \operatorname{Ker} A$.

Finding a basis for $\operatorname{Ker} A$:

We knew $v = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ is in coimg A, so $z = \begin{pmatrix}1\\-1\end{pmatrix}t$ is in ker A.

Then $x^* + z = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} t \\ -t \end{pmatrix}$, $t \in \mathbb{R}$. #