Lecture 4: Quick review from previous lecture

- We have discussed how to find LU factorization for a matrix A has n nonzero pivots :
  - either by elementary row operator type 1 (adding/subtracting one row to another row);
  - or type 2 (permutation)

  pivoting
- A is nonsingular $\Leftrightarrow$ A has a permuted LU factorization: $PA = LU$

Today we will discuss the **inverse** of a matrix.

- Homework 1 will be due this Friday.

1.5 Matrix Inverse

The inverse of a matrix is analogous to $a^{-1} = \frac{1}{a}$ of a scalar $a \neq 0$.
Thus, for [5] 1 by 1 matrix, it has inverse $[\,\frac{1}{5}\,]$. Then

$$[5][\,\frac{1}{5}\,] = [1].$$

- A is a square matrix, then its **inverse** A^{-1} is the $n \times n$ matrix satisfying

$$AA^{-1} = I_n = A^{-1}A.$$

Note that "Not every matrix has an inverse !!!"

$A = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}$. There is no matrix X with size 2×2 satisfying $XA = I_2$ or $AX = I_2$

In fact, we will see a square matrix has an inverse when it is nonsingular.

- We've already seen how to find the inverse of elementary row matrices:
  Example. The inverse of the matrix

$$E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is simply the matrix

$$E^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example. A permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad PP = I_3.$$

Thus,
$P^{-1} = P$.

- In general, however, finding A^{-1} will not be so easy. We will see a systematic method for doing so in the next class, known as *Gauss-Jordan elimination.*
- In the following, we will discuss 3 key facts:

> **Fact 1.** If the inverse of a matrix exists, then this inverse matrix is unique. In other words, if B and C are both inverse of A, then
>
> $$B = C.$$

Since B, C are inverse of A, $AB = I = BA$.
$AC = I = CA$.

$$B = BI = B(AC) = (BA)C = IC = C. \quad \#$$

> **Fact 2.** the inverse of the inverse is the original matrix. More precisely, $(A^{-1})^{-1} = A$.

Proof. This is an immediate consequence of the defining property of A^{-1}.

$$A^{-1}A = I. \Rightarrow (A^{-1})^{-1} = A.$$

- Continue the 3 key facts:

> **Fact 3.** If A and B are two invertible n-by-n matrices, then their product AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

$\underline{X} = B^{-1}A^{-1}$.

(1) $\underline{X}AB = (B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

(2) $AB\underline{X} = AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

Thus the inverse of AB is $B^{-1}A^{-1} = (AB)^{-1}$.

Remark: In general,

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$

§ The inverse of a 2×2 matrix.

We will find a simple formula of the inverse of 2×2 matrix.

Consider a general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let's compute its inverse $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ if it exists.

(1) $AX = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{cases} ax + bz = 1 \\ ay + bw = 0 \\ cx + dz = 0 \\ cy + dw = 1 \end{cases} \Rightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Case 1: $a \neq 0$

$$\xrightarrow{③ - \frac{c}{a}①} \left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & d - \frac{bc}{a} & 0 & -c/a \\ 0 & c & 0 & d & 1 \end{array}\right)$$

$$\xrightarrow{④ - \frac{c}{a}②} \left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ 0 & 0 & d - \frac{bc}{a} & 0 & -c/a \\ 0 & 0 & 0 & d - \frac{bc}{a} & 1 \end{array}\right).$$

$$w = \frac{a}{ad-bc} \quad \text{if} \quad ad - bc \neq 0.$$

A has no inverse if $ad - bc = 0$

$$x = \frac{d}{ad-bc}, \quad y = \frac{-b}{ad-bc}, \quad z = \frac{-c}{ad-bc}.$$

[Continue the computation:]

Case 2: $a = 0$, $c \neq 0$. Similar computation gives the same formula for x, y, z, w. if $ad - bc \neq 0$.

Case 3: $a = 0 = c$.

A has NO inverse.

(2) $XA = I_2$. Following a similar computation as above. #

Remark:

- If $ad - bc \neq 0$, then the 2-by-2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- The number "$ad - bc$" is known as the determinant of A, denoted by

$$\det(A) = ad - bc.$$

- In general, the determinant $\det(A)$ can be defined for a square matrix A of any size [Will discussed in later lectures], and

$$A \text{ is invertible if and only if } \det(A) \neq 0.$$