Lecture 5: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix A, that is, if A is a square matrix, then its **inverse** A^{-1} is the $n \times n$ matrix satisfying

$$AA^{-1} = I_n = A^{-1}A.$$

Today we will discuss the Gauss-Jordan Elimination to find the inverse of a general square matrix.

1.5 Matrix Inverse (Continue ...)

1 Introduction to Gauss-Jordan Elimination.

Gauss-Jordan Elimination is a similar process as Gaussian elimination and it also involves performing row operations to the matrix A.

Recall: In Gaussian elimination, the elementary row operations we used are

(1) Adding a multiple of one row to another row;

(2) switching the order of rows.

Now for Gauss-Jordan Elimination, in addition to row operators (1) and (2) above, we will use the 3^{rd} elementary row operator, that is,

(3) scaling a row of A by a nonzero multiple.

Note that "In the linear systems, multiplying one equation by a non-zero number obviously does not change the solution to the system."

Example: Let A be the 3-by-4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

- Then multiplying the second row by 8 results (that is, $8\times$ row 2) in the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} a & b & c & d \\ 8e & 8f & 8g & 8h \\ i & j & k & l \end{pmatrix}$$

- Like the other elementary row operations, row multiplication is realized by left multiplication with a specially chosen matrix, which is again formed by performing the desired row operation to the **identity matrix**.
- In this case, the matrix that associated to the "scales the second row by 8 is":

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$EA = \begin{pmatrix} a & b & c & d \\ 8e & 8f & 8g & 8h \\ i & j & k & l \end{pmatrix}$$

2 How to perform Gauss-Jordan elimination?

The goal is to find the matrix X satisfying $AX = XA = I$.

Before we start, we state a fact:

$\to AX = I$

Let A is a square $(n \times n)$ matrix, if X is the right inverse of A, then such X is automatically be the left inverse of A. $\to XA = I$.

In other words, a right inverse of a square matrix is automatically a left inverse, and conversely.

Generally, $A = A_{m\times n}$, (1) we say $B = B_{n\times m}$ is the left inverse of A if $BA = I_n$.

(2) we say $C = C_{n\times m}$ is the right inverse of A if $AC = I_m$.

- The Gauss-Jordan Elimination is to perform elementary row operations:

  E_i (1) adding a multiple of one row to another row;

  E_j (2) switching the order of rows;

  E_k (3) scaling a row of A by a nonzero multiple.

  to A to

  $$\text{turn } A \quad \text{into } I \text{ (the identity matrix),}$$

  if that is possible.

- Then we would have

$$(E_m E_{m-1} \cdots E_2 E_1)A = I.$$

- In other words, the product of all the elementary matrices $E_m E_{m-1} \cdots E_2 E_1$ is the inverse of A, that is,

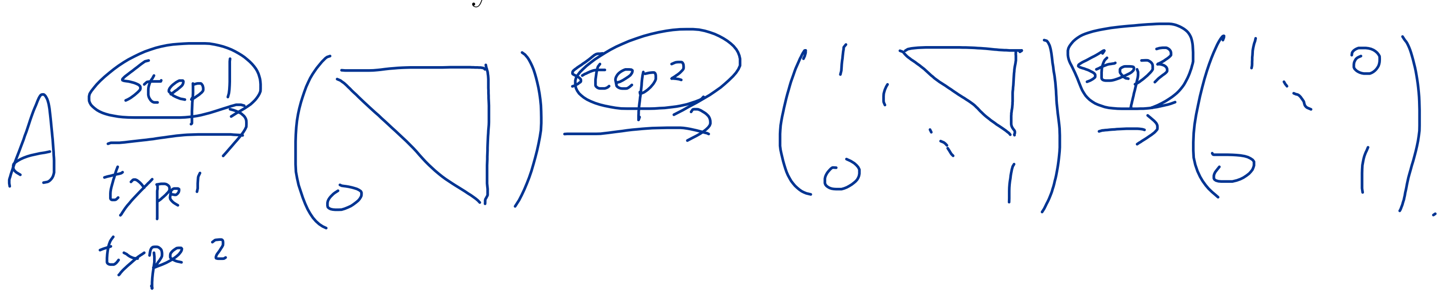
$$A^{-1} = E_m E_{m-1} \cdots E_2 E_1.$$

2.1 The operations to convert A to I are broken into 3 stages.

(1) bring $A \to$ upper triangular form;

(2) divide each row of A by the corresponding pivot (i.e. that row's diagonal element)

(3) More row operations to clear out the elements above the diagonal of A, and turn it into the identity.



Example. Find the inverse A^{-1} of

$$A = \begin{pmatrix} 0 & 4 & -2 \\ -1 & -3 & 4 \\ 2 & -6 & 6 \end{pmatrix}$$

$A \underline{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

$A(\vec{v}_1\ \vec{v}_2\ \vec{v}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Finding $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

augmented matrix

Step 1:

$$\left(\begin{array}{ccc|ccc} 0 & 4 & -2 & 1 & 0 & 0 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 2 & -6 & 6 & 0 & 0 & 1 \end{array}\right).$$

$$\xrightarrow[\text{①,②}]{\text{switch}} \left(\begin{array}{ccc|ccc} -1 & -3 & 4 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 2 & -6 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{③+2①} \left(\begin{array}{ccc|ccc} -1 & -3 & 4 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 0 & -12 & 14 & 0 & 2 & 1 \end{array}\right).$$

$$\xrightarrow{③+3②} \left(\begin{array}{ccc|ccc} -1 & -3 & 4 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 0 & 0 & 8 & 3 & 2 & 1 \end{array}\right),$$ we have finished step 1.

Step 2: Divide each row of A by it pivot.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -4 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right).$$

[Example continue...]

Step3 : clear out all elements above main diagonal.

$$\xrightarrow[\text{②}+2\text{③}]{\text{①}+4\text{③}} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 3/2 & 0 & 1/2 \\ 0 & 1 & 0 & 7/16 & 1/8 & 1/16 \\ 0 & 0 & 1 & 3/8 & 1/4 & 1/8 \end{array}\right)$$

$$\xrightarrow{\text{①}-3\text{②}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/16 & -3/8 & 5/16 \\ 0 & 1 & 0 & 7/16 & 1/8 & 1/16 \\ 0 & 0 & 1 & 3/8 & 1/4 & 1/8 \end{array}\right)$$

Thus, $A^{-1} = \begin{pmatrix} 3/16 & -3/8 & 5/16 \\ 7/16 & 1/8 & 1/16 \\ 3/8 & 1/4 & 1/8 \end{pmatrix}$ is the inverse of A.

RK: A^{-1} can be used to solve $Ax = b$.

EX: $b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. $x = A^{-1} b = \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. #

- A few additional comments; for details, refer to §1.5 in the textbook
- A triangular matrix is nonsingular if and only if all of its diagonal elements are non-zero; see page 39 in the book.
- Any lower triangular matrix with all non-zero diagonal elements has a lower triangular inverse, and any lower unitriangular matrix has a lower unitriangular inverse. Ditto if "lower" is replaced with "upper". Again, see page 39.

2.2 Turn to diagonal matrices.

Let $D = \mathrm{diag}(d_1, \ldots, d_m)$ is an m-by-m diagonal matrix.

- DA is equal to A with the i^{th} row scaled by d_i.

EX: $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $A = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$DA = \begin{pmatrix} 4a \\ 2b \\ 3c \end{pmatrix} \quad \#$$

- D is invertible if all of its diagonal entries are non-zero.

$$D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

Check $D^{-1}D = I_3$

$DD^{-1} = I_3$.

- Let D_1 and D_2 be 2 diagonal matrices. Then so is D_1D_2.

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1}b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_nb_n \end{pmatrix}$$