Lecture 5: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix $A$, that is, if $A$ is a square matrix, then its inverse $A^{-1}$ is the $n \times n$ matrix satisfying

$$AA^{-1} = I_n = A^{-1}A.$$ 

Today we will discuss the Gauss-Jordan Elimination to find the inverse of a general square matrix.
1 Introduction to Gauss-Jordan Elimination.

Gauss-Jordan Elimination is a similar process as Gaussian elimination and it also involves performing row operations to the matrix $A$.

Recall: In Gaussian elimination, the elementary row operations we used are

(1) Adding a multiple of one row to another row;
(2) switching the order of rows.

Now for Gauss-Jordan Elimination, in addition to row operators (1) and (2) above, we will use the 3\textsuperscript{rd} elementary row operator, that is,

(3) scaling a row of $A$ by a nonzero multiple.

Note that “In the linear systems, multiplying one equation by a non-zero number obviously does not change the solution to the system.”

Example: Let $A$ be the 3-by-4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

• Then multiplying the second row by 8 results (that is, $8 \times$ row 2) in the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} a & b & c & d \\ 8e & 8f & 8g & 8h \\ i & j & k & l \end{pmatrix}$$
• Like the other elementary row operations, row multiplication is realized by left multiplication with a specially chosen matrix, which is again formed by performing the desired row operation to the identity matrix.

• In this case, the matrix that associated to the “scales the second row by 8 is”:

\[
E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then

\[
EA = \begin{pmatrix} a & b & c & d \\ 8e & 8f & 8g & 8h \\ i & j & k & l \end{pmatrix}
\]

2 How to perform Gauss-Jordan elimination?

The goal is to find the matrix \( X \) satisfying \( AX = XA = I \).

Before we start, we state a fact:

Let \( A \) is a square \((n \times n)\) matrix, if \( X \) is the right inverse of \( A \), then such \( X \) is automatically be the left inverse of \( A \).

In other words, a right inverse of a square matrix is automatically a left inverse, and conversely.

Generally, \( A = A_{m \times n} \), we say \( B = B_{n \times m} \) is the left inverse of \( A \) if \( BA = In \).

\(2\) we say \( C = C_{n \times m} \) is the right inverse of \( A \) if \( AC = Im \).
The Gauss-Jordan Elimination is to perform elementary row operations:

1. adding a multiple of one row to another row;
2. switching the order of rows;
3. scaling a row of \( A \) by a nonzero multiple.

to \( A \) to turn \( A \) into \( I \) (the identity matrix), if that is possible.

Then we would have

\[
(E_mE_{m-1} \cdots E_2E_1)A = I.
\]

In other words, the product of all the elementary matrices \( E_mE_{m-1} \cdots E_2E_1 \) is the inverse of \( A \), that is,

\[
A^{-1} = E_mE_{m-1} \cdots E_2E_1.
\]

2.1 **The operations to convert \( A \) to \( I \) are broken into 3 stages.**

1. bring \( A \) → upper triangular form;
2. divide each row of \( A \) by the corresponding pivot (i.e. that row’s diagonal element)
3. More row operations to clear out the elements above the diagonal of \( A \), and turn it into the identity.
Example. Find the inverse $A^{-1}$ of

$$A = \begin{pmatrix} 0 & 4 & -2 \\ -1 & -3 & 4 \\ 2 & -6 & 6 \end{pmatrix}.$$ 

$$AX = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$A\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$ Finding $\vec{v}_1, \vec{v}_2, \vec{v}_3.$

Step 1: Augmented matrix

$$\begin{pmatrix} 0 & 4 & -2 & | & 1 & 0 & 0 \\ -1 & -3 & 4 & | & 0 & 1 & 0 \\ 2 & -6 & 6 & | & 0 & 0 & 1 \end{pmatrix}.$$ 

Switch

$$\begin{pmatrix} 1 & 1 & 4 & | & 0 & 1 & 0 \\ 0 & 4 & -2 & | & 1 & 0 & 0 \\ 2 & -6 & 6 & | & 0 & 0 & 1 \end{pmatrix}.$$ 

\[2 \times 1 \rightarrow \]

$$\begin{pmatrix} -1 & -3 & 4 & | & 1 & 0 & 0 \\ 0 & 4 & -2 & | & 1 & 0 & 0 \\ 0 & -12 & 14 & | & 0 & 2 & 1 \end{pmatrix}.$$ 

\[3 \times 2 \rightarrow \]

$$\begin{pmatrix} -1 & -3 & 4 & | & 0 & 1 & 0 \\ 0 & 4 & -2 & | & 1 & 0 & 0 \\ 0 & 0 & 8 & | & 0 & 8 & 1 \end{pmatrix}.$$ We have finished step 1.

Step 2: Divide each row of $A$ by its pivot.

$$\begin{pmatrix} 1 & 3 & -4 & | & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & | & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & | & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}.$$
Step 3: clear out all elements above main diagonal.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 & 1/2 \\n7/6 & 1/8 & 7/6 \\
3/8 & 1/4 & 1/8
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
3/16 & -3/8 & 5/16 \\
7/6 & 1/8 & 5/16 \\
3/8 & 1/4 & 1/8
\end{pmatrix}
\]

Thus, \( A^{-1} = \begin{pmatrix}
3/16 & -3/8 & 5/16 \\
7/6 & 1/8 & 5/16 \\
3/8 & 1/4 & 1/8
\end{pmatrix}\) is the inverse of \( A \).

**RK:** \( A^{-1} \) can be used to solve \( Ax = b \).

**EX:** \( b = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \), \( x = A^{-1} b = \begin{pmatrix} \frac{1}{2} \\ -1/2 \end{pmatrix} \).

- A few additional comments; for details, refer to §1.5 in the textbook
- A triangular matrix is nonsingular if and only if all of its diagonal elements are non-zero; see page 39 in the book.
- Any lower triangular matrix with all non-zero diagonal elements has a lower triangular inverse, and any lower unitriangular matrix has a lower unitriangular inverse. Ditto if “lower” is replaced with “upper”. Again, see page 39.
2.2 Turn to diagonal matrices.

Let $D = \text{diag}(d_1, \ldots, d_m)$ is an $m$-by-$m$ diagonal matrix.

- $DA$ is equal to $A$ with the $i^{th}$ row scaled by $d_i$.

  $\begin{bmatrix}
  a \\
  b \\
  c \\
  \end{bmatrix}
  \begin{bmatrix}
  4 & 0 & 0 \\
  0 & 2 & 0 \\
  0 & 0 & 3 \\
  \end{bmatrix}
  = 
  \begin{bmatrix}
  4a \\
  2b \\
  3c \\
  \end{bmatrix}$

- $D$ is invertible if all of its diagonal entries are non-zero.

  $D^{-1} = 
  \begin{bmatrix}
  \frac{1}{4} & 0 & 0 \\
  0 & \frac{1}{2} & 0 \\
  0 & 0 & \frac{1}{3} \\
  \end{bmatrix}
  \quad \text{Check} \quad D^{-1}D = I_3$

- Let $D_1$ and $D_2$ be 2 diagonal matrices. Then so is $D_1D_2$.

  $\begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_n \\
  \end{bmatrix}
  \begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n \\
  \end{bmatrix}
  = 
  \begin{bmatrix}
  a_1b_1 \\
  a_2b_2 \\
  \vdots \\
  a_nb_n \\
  \end{bmatrix}$