Lecture 6: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix $A$, that is, if $A$ is a square matrix, then its inverse $A^{-1}$ is the $n \times n$ matrix satisfying
  \[ AA^{-1} = I_n = A^{-1}A. \]

- Gauss-Jordan Elimination to find the inverse of a general square matrix.

Today we will discuss $LDV$ factorization for regular square matrices, the transpose, and General linear system.

- Quiz 2 (covers sec. 1.4-1.6) will take place in the beginning of the class on Wed. 2/12
§ Turn to diagonal matrices.

Let $D = \text{diag}(d_1, \ldots, d_m)$ is an $m$-by-$m$ diagonal matrix.

- $DA$ is equal to $A$ with the $i^{th}$ row scaled by $d_i$.

\[
D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.
\]

\[
DA = \begin{pmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{pmatrix}
\]

- $D$ is invertible if all of its diagonal entries are non-zero.

\[
D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad D^{-1} D = I_3, \quad DD^{-1} = I_3
\]

- Let $D_1$ and $D_2$ be 2 diagonal matrices. Then so is $D_1D_2$.

\[
\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1}b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_nb_n \end{pmatrix}
\]
§ LDV factorization.
If $A$ is a regular matrix, we already know we can write $A = LU$, where $U$ is upper triangular with non-zero diagonal elements (the pivots), and $L$ is lower triangular with all diagonal elements equal to 1.

- When a triangular matrix has all 1's on its diagonal, we say it is unitriangular. So the $L$ in the $LU$ decomposition is lower unitriangular.

- Turn $U$ into

$$U = DV$$

where $D$ is diagonal matrix and matrix $V$ is upper unitriangular.

\[ U = \begin{pmatrix} 3 & 4 & 5 \\ 0 & -2 & 1 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \]

Thus, $A = LU = LDV$.

**Fact:** If $A$ is nonsingular (but not necessarily regular), we can form the permuted LDV factorization

$$PA = LDV,$$

where $P$ is a permutation matrix, $L$ is lower unitriangular, $D$ is diagonal, and $V$ is upper unitriangular.
1.6 Transposes and Symmetric Matrices

- If $A$ is a matrix of any dimensions, then its **transpose**, denoted $A^T$, is the matrix that switches the roles of the rows and columns of $A$.

  \[ V = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad V^T = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad W = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad W^T = \begin{pmatrix} 1 & 4 & 5 \end{pmatrix} \]

- If $a_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are the elements of $A$, then element $(i, j)$ of $A^T$ is $a_{j,i}$.

- In particular, if $A$ is an $m$-by-$n$ matrix, then $A^T$ is an $n$-by-$m$ matrix.

- For example, if

  \[ A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}_{4 \times 2} \]

  then

  \[ A^T = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}_{2 \times 4} \]

  \[ (A^T)^T = A \]

- Here are some basic properties of the transposition operations.

- First, taking the transpose twice returns you to the original matrix:

  \[ (A^T)^T = A \]

- Transposition is compatible with matrix addition and scalar multiplication. If $A$ and $B$ are matrices of the same dimensions and $c$ is a scalar, then:

  \[ (A + B)^T = A^T + B^T \]

  and

  \[ (cA)^T = cA^T \]
• When you transpose a product of matrices, it is the product of the transposes in the opposite order:

\[(AB)^T = B^T A^T\]

\[A: m \times n \text{ matrix}, \quad B: n \times p \text{ matrix} \implies AB \quad m \times p \implies (AB)^T \quad p \times m.\]

\[A^T: n \times m, \quad B^T: p \times n \text{ matrix} \implies B^T A^T \quad p \times m.\]

*This property can be checked directly from the definition of matrix multiplication.

• Transposition commutes with inversion: \((A^T)^{-1} = (A^{-1})^T.\)

\[X = (A^{-1})^T.\]

\[X A^T = (A^{-1})^T A^T = (A A^{-1})^T = I^T = I.\]

*Similarly, \(A^T X = I.\) Thus, \(X = (A^T)^{-1}.\)

• A matrix is said to be **symmetric** if it is equal to its own transpose.

\[A^T = A\]

For example, the matrix

\[A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \quad A^T = A.\]

is symmetric.

• By necessity, symmetric matrices must be square.
**Fact.** A symmetric matrix $A$ is regular if and only if $A = LDL^T$, where $L$ is a lower unitriangular matrix and $D$ is a diagonal matrix with nonzero diagonal entries.

Since $A$ is regular, $A = LDU$.

Since $A$ is symmetric, $A^T = A$.

Thus, $A = LDU = V^TDL^T$.

Then $A = LDV = V^TDL^T$.

Since LDV factorization is unique, $L = V^T$.

Thus, $A = LDV = LDLD^T$.

- For any nonsingular matrix $A$, we can use the decomposition $PA = LDV$ to solve a linear system $Ax = b$. First solve $Ly = Pb$, then $Dz = y$, then $Vx = z$.

- Using the $LDV$ or $LDLT$ decompositions is typically not any easier than using $LU$ for solving linear systems. But writing the symmetric version $LDLT$ can have some advantages we will see later in the semester.
1.8 General linear system

A may be rectangular, or A may be square but can have zero pivots after being brought to upper triangular form.

Let’s look at the following different situations:

- the number of variables > the number of equations:

**Example 1.** Solve the linear system:

\[
\begin{pmatrix}
1 & 2 & 3 \\
-2 & 2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
4 \\
5
\end{pmatrix}
\]

**Gaussian Elimination**

\[\begin{pmatrix}
1 & 2 & 3 \\
0 & 6 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
4 \\
13
\end{pmatrix}
\]

2nd equation: \(6y + 2z = 13\).

Fix \(z\), we write \(y\) in terms of \(z\), that is

\[y = \frac{13 - 2z}{6}.
\]

1st equation: \(x + 2y + 3z = 4\). Solve \(x\) in terms of \(z\),

\[x = 4 - 2y - 3z = 4 - 2 \left( \frac{13 - 2z}{6} \right) - 3z = 4 - \frac{13 - 2z}{3} - 3z = \frac{12 - 13 + 2z - 9z}{3} = \frac{-1 - 7z}{3}.
\]

Solution of \(Ax = b\) is

\[\begin{pmatrix}
\frac{-1 - 7z}{3} \\
\frac{-13 - 2z}{6} \\
z
\end{pmatrix}
\] for any number \(z\).