Lecture 6: Quick review from previous lecture

- We talked about what is the inverse of a given square matrix A, that is, if A is a square matrix, then its **inverse** A^{-1} is the $n \times n$ matrix satisfying

$$AA^{-1} = I_n = A^{-1}A.$$

- Gauss-Jordan Elimination to find the inverse of a general square matrix.

Today we will discuss LDV factorization for regular square matrices, the transpose, and General linear system.

- Quiz 2 (covers sec. 1.4-1.6) will take place in the beginning of the class on Wed. 2/12

§ Turn to diagonal matrices.

Let $D = \mathrm{diag}(d_1, \ldots, d_m)$ is an m-by-m diagonal matrix.

- DA is equal to A with the i^{th} row scaled by d_i.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

$$DA = \begin{pmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{pmatrix}$$

- D is invertible if all of its diagonal entries are non-zero.

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \qquad D^{-1}D = I_3, \quad DD^{-1} = I_3$$

- Let D_1 and D_2 be 2 diagonal matrices. Then so is $D_1 D_2$.

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1}b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_nb_n \end{pmatrix}$$

§ LDV factorization.

lower unitriangular

If A is a regular matrix, we already know we can write $A = LU$, where U is upper triangular with non-zero diagonal elements (the pivots), and L is lower triangular with all diagonal elements equal to 1.

- When a triangular matrix has all 1's on its diagonal, we say it is **unitriangular**. So the L in the LU decomposition is lower unitriangular.

- Turn U into

$$U = DV$$

where D is diagonal matrix and matrix V is upper unitriangular.

EX: $U = \begin{pmatrix} 3 & 4 & -5 \\ 0 & -2 & 1 \\ 0 & 0 & 7 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 4/3 & -5/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}}_{V}.$

Thus, $A = LU = LDV.$

PA is regular

> **Fact:** If A is nonsingular (but not necessarily regular), we can form the permuted LDV factorization
> $$PA = LDV,$$
> where P is a permutation matrix, L is lower unitriangular, D is diagonal, and V is upper unitriangular.

1.6 Transposes and Symmetric Matrices

- If A is a matrix of any dimensions, then its **transpose**, denoted A^T, is the matrix that switches the roles of the rows and columns of A.

$v = (1\ 2\ 3)$, $v^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$, $w^T = (1\ 4\ 5)$.

- If $a_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are the elements of A, then element (i, j) of A^T is $a_{j,i}$.

- In particular, if A is an m-by-n matrix, then A^T is an n-by-m matrix.

- For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}_{4\times 2}$$

then

$$A^T = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}_{2\times 2}, \quad (A^T)^T = A.$$

- Here are some basic properties of the transposition operations.

- First, taking the transpose twice returns you to the original matrix:

$$(A^T)^T = A$$

- Transposition is compatible with matrix addition and scalar multiplication. If A and B are matrices of the same dimensions and c is a scalar, then:

$$(A + B)^T = A^T + B^T$$

and

$$(cA)^T = cA^T$$

- When you transpose a product of matrices, it is the product of the transposes *in the opposite order*:

$$(AB)^T = B^T A^T$$

A: $m\times n$ matrix, B: $n\times p$ matrix $\Rightarrow$ AB $m\times p$ $\Rightarrow$ $(AB)^T$ $\underline{p\times m}$.

A^T: $n\times m$ ", B^T: $p\times n$ matrix $\Rightarrow$ $B^T A^T$ $\underline{p\times m}$.

*This property can be checked directly from the definition of matrix multiplication.

- Transposition commutes with inversion: $(A^T)^{-1} = (A^{-1})^T$.

$X = (A^{-1})^T$.

$X A^T = (A^{-1})^T A^T = (A A^{-1})^T = I^T = I$.

Similar, $A^T X = I$. Thus, $X = (A^T)^{-1}$. #

- A matrix is said to be **symmetric** if it is equal to its own transpose.

$$\boxed{A^T = A}$$

For example, the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$, $A^T = A$.

is symmetric.

- By necessity, symmetric matrices must be square.

Fact. A symmetric matrix A is regular if and only if $A = LDL^T$, where L is a lower unitriangular matrix and D is a diagonal matrix with nonzero diagonal entries.

Since A is regular, $A = LDV$.

Since A is symmetric, $A^T = A$

$A^T = (LDV)^T = V^T D^T L^T = V^T D L^T$, $A = LDV$

Then $A = LDV = \underbrace{V^T}_{\text{lower unitriangular}} D \underbrace{L^T}_{\text{upper unitriangular}}$

Since LDV factorization is unique, $L = V^T$, $\boxed{V = L^T}$

Thus, $A = LDV = LDL^T$. #

$PAx = Pb \quad L\overbrace{D\underbrace{Vx}_{z}}^{y} = Pb.$

$$\begin{cases} Ly = Pb \\ Dz = y \\ Vx = z \end{cases}$$

- For any nonsingular matrix A, we can use the decomposition $\boxed{PA = LDV}$ to solve a linear system $A\mathbf{x} = \mathbf{b}$. First solve $L\mathbf{y} = P\mathbf{b}$, then $D\mathbf{z} = \mathbf{y}$, then $V\mathbf{x} = \mathbf{z}$.
- Using the LDV or LDL^T decompositions is typically not any easier than using LU for solving linear systems. But writing the symmetric version LDL^T can have some advantages we will see later in the semester.

1.8 General linear system

A may be rectangular, or A may be square but can have zero pivots after being brought to upper triangular form.

Let's look at the following different situations:

- the number of variables > the number of equations:

  Example 1. Solve the linear system:

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Gaussian Elimination

$$\xrightarrow{②+2①} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \end{pmatrix}$$

pivots

2^{nd} equation: $6y + 2z = 13$.

Fix z, we write y in terms of z, that is

$$y = \frac{13-2z}{6}.$$

1^{st} equation: $x + 2y + 3z = 4$. Solve x in terms of z.

$$\begin{aligned} x &= 4 - 2y - 3z \\ &= 4 - 2\left(\frac{13-2z}{6}\right) - 3z \\ &= 4 - \frac{13-2z}{3} - 3z = \frac{12-13+2z-9z}{3} \\ &= \frac{-1-7z}{3}. \end{aligned}$$

Solution of $Ax=b$ is

$$\left(\frac{-1-7z}{3}, \frac{13-2z}{6}, z\right)^T, \text{ for any number } z.$$