Lecture 7: Quick review from previous lecture

- We saw what is the transpose of a matrix and some of its properties.
- Once in this "staircase" shape, we can solve a linear system with this coefficient matrix from bottom to top, or determine if there is no solution.
- Quiz 2 (covers sec. 1.4-1.6) will take place in the beginning of the class on Wed. 2/12

1.8 General Linear system (Continue...)

- the number of variables $\leq$ the number of equations:

Example 2. Solve the linear system:

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

$$\xrightarrow[\text{③}-2\text{①}]{\text{②}-3\text{①}} \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix}$$

$$\xrightarrow{\text{③}+\text{②}} \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ -12 \end{pmatrix},$$

Or

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 4 & -2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & -10 \\ 0 & 0 & -12 \end{array}\right)$$

3rd equ.: $0 \neq -12$. Thus this system has NO solution.

Example 3. Solve the linear system:

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 18 \end{pmatrix}$$

By performing the same process as above, we have

$$\begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 0 \end{pmatrix}.$$

3rd equ: $0 = 0$.

2nd equ: $-4y = -10$. $y = 5/2$.

1st equ: $x + 2y = 4$, $x = 4 - 5 = -1$

$(x, y)^T = (-1, 5/2)^T$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5/2 \end{pmatrix}$. #

§ Row echelon form

- Gaussian elimination (with pivoting) can bring any matrix to the following form, which is called row echelon form:

$$
\overbrace{\left(\begin{array}{cccccccccccccccc}
\circledast & * & \dots & * & * & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & \circledast\ \text{pivot} & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast\ \text{pivot} & \dots & \dots & * & * & * & \dots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast\ \text{pivot} & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{array}\right)}^{n}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} m
$$

pivot

- Once in this "staircase" shape, we can solve a linear system with this coefficient matrix from bottom to top, as we just did.

- The circled values are called the pivots.

- The number of pivots is called the rank of the matrix A.

$$\boxed{rank(A) = \text{number of its pivots}}$$

- (Will show later) If a matrix is brought to two different row echelon forms, the ranks are the same.
  In other words, the rank depends only on the matrix, not the particular choice of row operations we used to bring it to row echelon form.

Remark:

$A_{n\times n} \to [\nabla]_{n\times n}$.

1. As we've seen, any square matrix A can be brought to upper triangular form, which is a special case of row echelon form.

2. When A is nonsingular, all the diagonal elements of the upper triangular matrix will be nonzero, and so the rank of $A = A_{n\times n}$ is n.

3. Conversely, if the rank of $A_{n\times n}$ is n, then A is nonsingular. Another way of saying this is that nonsingular matrices are "full rank", since they have the maximum allowed rank.

$A_{n\times n}$ is nonsingular $\iff$ $\text{rank}(A) = n$.

Some definitions:

- When solving a general linear system, the variables that correspond to columns not containing a pivot can be chosen arbitrarily. These are called free variables.

- The variables corresponding to columns that do contain a pivot are called basic variables.

  *We solve for the basic variables in terms of the free variables.

- If A is in row echelon form and has rank r, then only the first r rows of A are non-zero. For the system to have a solution, the right hand side must be zero in the last $m - r$ entries.

  When this occurs, we say that the system is compatible. Phrased differently, a compatible system has <u>at least one solution</u>.

  * Note that the compatibility of a system $A\mathbf{x} = \mathbf{b}$ depends both on the coefficient matrix A and the right hand side $\mathbf{b}$.

Summary:

1. A system may have 0, 1 or infinitely many solutions, but no other numbers. So if there are two solutions, then there must be infinitely many solutions.

has solution(s)

2. Let A be a $m \times n$ matrix. When the system $A\mathbf{x} = \mathbf{b}$ is compatible and

$$rank(A) = \text{number of variables n},$$

there is exactly 1 solution.

3. Having a unique solution is only possible if $m \geq n$ (i.e. for square or tall coefficient matrices).

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} & \\ & \\ & \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

4. When $n > m$ (i.e. the coefficient matrix is short and wide), there are either 0 or infinitely many solutions.

EX: Solve the linear system

$$\text{pivot} \leftarrow \begin{pmatrix} 1 & -5 & -1 & -3 \\ 0 & 3 & -2 & 2 \\ 0 & -3 & 2 & -3 \end{pmatrix}_{3\times 4} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$$

$$\xrightarrow{③+②} \begin{pmatrix} 1 & -5 & -1 & -3 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix}.$$

3rd eqn: $w = 1$.

2nd eqn: $3\underbrace{y}_{\text{basic variable}} - 2\overbrace{z}^{\text{free variable}} + 2\overbrace{w}^{} = -4$. Solve y in terms of z.

$$y = \frac{-6+2z}{3}.$$

1st eqn: $x - 5y - z - 3w = -5$.

$$x = -5 + 5\left(\frac{-6+2z}{3}\right) + z + 3.$$

$$= \frac{13z - 36}{3}.$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{13z-36}{3} \\ \frac{-6+2z}{3} \\ z \\ 1 \end{pmatrix} \quad \#$$

Recall that the rank r of a matrix is the number of rows that are not identically zero, after the matrix has been brought to row echelon form.

Fact. Let A be $m \times n$ matrix. then

$$0 \leq r = rank(A) \leq min\{m, n\}$$

By definition, $r \leq m$.

$r \leq n$, since the row echelon form is staircase.

§ Homogeneous Systems.

When the right hand side of a linear system is the 0 vector, we say the system is homogeneous. That is, a homogeneous system is of the form $A\mathbf{x} = \mathbf{0}$.

- The vector $\mathbf{x} = \mathbf{0}$ is always a solution to this system, since $A\mathbf{0} = \mathbf{0}$.
- If the matrix A is nonsingular, then $\mathbf{x} = \mathbf{0}$ is the unique solution.
- For other matrices A, there may be many vectors $\mathbf{x}$ with $A\mathbf{x} = \mathbf{0}$.