Lecture 8: Quick review from previous lecture

- Gaussian elimination (with pivoting) can bring any matrix to the following form, which is called row echelon form:

$$
\begin{pmatrix}
\circledast & * & \dots & * & * & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & \circledast & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast & \dots & \dots & * & * & * & \dots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}
$$

- The number of pivots is called the rank of the matrix A.

$$\mathsf{rank}(A) = \text{number of its pivots}$$

- $n \times n$ matrix A is nonsingular if and only if $\mathsf{rank}(A) = n$.
- This system $A\mathbf{x} = \mathbf{0}$ is called homogeneous.

Today we will discuss the determinant and vector space.

- Quiz 2 (covers sec. 1.4-1.6) will take place in the beginning of the class on Wed. 2/12

Example.

$$\begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -4 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Fact. Let $A = A_{m\times n}$ be the matrix of size $m \times n$.

(1)

$$A\mathbf{x} = \mathbf{0} \text{ has a nontrivial solution } \mathbf{x} \neq \mathbf{0} \Leftrightarrow \ rank(A) < n.$$

(2) If $m < n$, the system $A\mathbf{x} = \mathbf{0}$ always has a nontrivial solution.

(3) If $m = n$, the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\Leftrightarrow$ A is singular.

- Later, we will learn that the set of all $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$ is called the **kernel** of A.

1.9 Determinants

Recall that

- We saw previously that a 2-by-2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse if and only if (iff) $\det(A) \neq 0$, where $\det(A) = ad - bc$ is the *determinant* of A.

- Thus,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is invertible} \quad \Leftrightarrow \quad \det(A) \neq 0 \quad \Leftrightarrow \quad \text{A is nonsingular}$$

Today we will now see how to generalize this to all square matrices.

§ Generalize to $n \times n$ matrix.

The key ingredient will be the permuted LU factorization that we have already seen.

As we have known, any square matrix A can be turn to LU form (Gaussian elimination with pivoting):

$$PA = LU$$

where P is a permutation matrix (the product of all the elementary row swap matrices), L is lower unitriangular, and U is upper triangular.

- We also know that A is invertible precisely when

$$U = \begin{pmatrix} u_{11} & & & \\ 0 & u_{22} & & \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix} \quad \text{with all } u_{11}, \ldots, u_{nn} \text{ are nonzero.}$$

- Thus, we conclude that the *product* of all these numbers,

$$\det(U) = u_{11} \cdots u_{nn} \neq 0 \quad \Leftrightarrow \quad A \text{ is invertible (nonsingular)}$$

- Motivated by this, we **define** $\det(A)$ as follows:

(1) If A is regular, then

$$\det(A) = \det(U) = \underbrace{\prod_{i=1}^{n} u_{i,i}}_{= u_{11} u_{22} \cdots u_{nn}}.$$

(2) If A is nonsingular, and requires k row interchanges to arrive at its permuted factorization $PA = LU$, then

$$\det(A) = \det(P)\det(U) = (-1)^k \prod_{i=1}^{n} u_{i,i}.$$

In summary, one has

$$\boxed{\det(A) = (-1)^k \prod_{i=1}^{n} u_{i,i}}$$

where k denotes the number of row permutations we performed to bring A into upper triangular form.

✓Then $\det(A) \neq 0$ iff A is nonsingular.

Example. Compute the determinant of

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -4 & -2 \end{pmatrix}$$

By Gaussian Elimination with pivoting,

$$\xrightarrow[\text{④}-\text{①}]{\text{②}-2\text{①}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & -3 & -4 \end{pmatrix}$$

$$\xrightarrow[\text{④}-\text{②}]{\text{③}-2\text{②}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

$$\xrightarrow[\text{③④}]{\text{Switch}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix} = U.$$

$$\det A = (-1)^1 \det U = -1(1\cdot 1\cdot(-2)\cdot 3)$$

$$= \underline{6} \ \#$$

RK: $\det A \neq 0$ so A is nonsingular.

An immediate result:

- Note that if $A_{n\times n}$ has a row consisting entirely of zeros, then $\det(A) = 0$.

EX: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}}_{=U}$, $\underset{\det U = 1\cdot(-3)\cdot 0}{\overset{\|}{}}$ $= 0$.

§ Elementary Row Operations on the determinant of a $n \times n$ matrix A:

- If B is a matrix obtained by adding a multiple of one row of A to another row of A. Then
$$\det(A) = \det(B).$$
- If B is a matrix obtained by interchanging any two rows of A, then
$$\det(B) = -\det(A).$$
- If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then
$$\det(B) = k\det(A).$$

EX: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $B = \begin{pmatrix} a & b & c \\ 3a+d & 3b+e & 3c+f \\ g & h & i \end{pmatrix}$

$\det A = \det B$

EX: $B = \begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}$, $\det B = 3\det A$

§ The det operator behaves well with matrix multiplication, inversion, and transposition (but NOT addition!).

(0)
$$\det(cA) = c^n \det(A).$$

(1) If A and B are two square matrices of the same size, then
$$\det(AB) = \det(A)\det(B).$$

*In particular, note that $\det(AB) = \det(BA)$, even though AB need not equal BA.

(2) AB is invertible $\Leftrightarrow$ both A and B are invertible.

(3) If A is invertible, then
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

(3) $1 = \det I = \det(A^{-1}A) \overset{(1)}{=} \det A^{-1} \det A$

$\Rightarrow \det A^{-1} = 1/\det A$

(4) $\det(A^T) = \det(A)$.

EX: $A = \begin{pmatrix} 2 & 4 \\ 0 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 12 \\ 0 & 21 \end{pmatrix}$.

$\det A = 2 \cdot 7$, $\det B = 6 \cdot 21 = 3^2 \det A$

(2) $\det(AB) \overset{(1)}{=} \det A \det B \neq 0$.

So $\det A \neq 0$ and $\det B \neq 0$, which gives A, B are invertible.

Similar for the converse statement.

§ There is **a formula for** $\det(A)$ that is explicit in its dependence on the entries of A, but is typically harder to compute.

Page 72 in the textbook: $n \times n$ matrix A with entries a_{ij}: $$\det(A) = \sum_{\substack{\text{permutations} \\ \pi \text{ of } \{1,\dots,n\}}} \operatorname{sign}(\pi) a_{\pi(i),1} a_{\pi(i),2} \cdots a_{\pi(i),n}$$

The sum is over all possible permutations π of the rows of A.

The sign of the permutation written $\operatorname{sign}(\pi)$, and $\operatorname{sign}(\pi) = \det(P)$, where P is the corresponding permutation matrix.

$$\operatorname{sign}(\pi) = \begin{cases} +1 & \text{if the permutation is composed of an "even" number of row interchange;} \\ -1 & \text{if the permutation is composed of an "odd" number of row interchange.} \end{cases}$$

EX: $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 1\, a_{1,1} a_{2,2} a_{3,3} - 1\, a_{2,1} a_{1,2} a_{3,3}$

$a_{2,1} a_{3,2} a_{1,3} - 1\, a_{3,1} a_{2,2} a_{1,3}$

$a_{3,1} a_{1,2} a_{2,3} - 1\, a_{1,1} a_{3,2} a_{2,3}$

*Unsurprisingly, this formula is typically NOT used to actually compute $\det(A)$, except in special cases. From our perspective, the main interest in this formula is that it depends only on A, not on the steps of Gaussian elimination.

*For most matrices, the "best" way to compute a determinant is "Gaussian Elimination with pivoting".