Lecture 9: Quick review from previous lecture

- Gaussian elimination (with pivoting) can bring any matrix to the following form, which is called row echelon form:

$$\begin{pmatrix}
\circledast & * & \dots & * & * & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & \circledast & \dots & * & * & \dots & \dots & * & * & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & \circledast & \dots & \dots & * & * & * & \dots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \circledast & * & \dots & * \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0
\end{pmatrix}$$

- The number of pivots is called the rank of the matrix A.

$$\mathsf{rank}(A) = \text{number of its pivots}$$

- $$\det(A) = (-1)^k \prod_{i=1}^{n} u_{i,i}$$

  where k denotes the number of row permutations we performed to bring A into upper triangular form.

Today we will discuss the vector space.

- Quiz 2 (covers sec. 1.4-1.6) will take place in the beginning of the class on Wed. 2/12

2.1 Real Vector Spaces

Definition: A **vector space** is a set V equipped with two operations:

(1) (Addition) If $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} + \mathbf{w} \in V$.

(2) (Scalar Multiplication) Multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.
(e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
(f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

Example 1. $\mathbb{R}^n$ is a vector space. with addition $(a_1, a_2, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$

Scalar $c(a_1, a_2, \ldots, a_n) = (ca_1, \ldots, ca_n)$

(c) zero element $\exists$ $(\underbrace{0, \ldots, 0}_{n \text{ components}})$: $\vec{a} + (0, \ldots, 0) = \vec{a} = (0, \ldots, 0) + \vec{a}$

(d) $\vec{a} + (-\vec{a}) = (0, \ldots, 0)$.

other conditions are true as well.

Example 2. The set of all $m \times n$ matrices (which we denote by $\mathsf{M}_{m\times n}(\mathbb{R})$) with entries from $\mathbb{R}$ is a "vector space" when equipped with the usual **matrix addition** and **scalar multiplication**.

For instance, $\mathsf{M}_{2\times 3}(\mathbb{R})$ is a vector space.

(addition) $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{m1} & & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \in \mathsf{M}_{m\times n}$

(scalar) $c\begin{pmatrix} & & \end{pmatrix} = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix} \in \mathsf{M}_{m\times n}$

Example 3. We consider the collection of all functions f defined on an interval $[a, b]$. We'll call this set $\mathcal{F} = \mathcal{F}([a, b])$. Like column vectors, we can add two functions together:

$$(f+g)(x) \overset{\text{(define)}}{=} f(x) + g(x)$$

and we can multiply a function by a scalar:

$$(c \cdot f)(x) \overset{\text{(define)}}{=} c \cdot f(x)$$

(1) $f, g \in \mathcal{F}$, $f+g \in \mathcal{F}$; (2) $cf \in \mathcal{F}$.

(a) — (g) : (a) $f+g = g+f$,
(b)
(c) zero element $f=0$. $0+g = g = g+0$.
⋮
(g) True.

$P^{(4)}$: $x+1$ (Yes)
$x^3 - 100000$ (Yes)
0 (Yes)
⋮

Example 4. Consider the space

$$\mathcal{P}^{(n)} = \{p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\}$$

consisting of all real polynomials of degree $\leq n$.

We have a way to define addition of two polynomials of degree $\leq n$:
If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, then

$$(p+q)(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

and

$$cp(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots + ca_1 x + ca_0.$$

(1) (2) hold. check (a) — (g) : (a) $p+q = q+p$
(b)
(c) zero element : 0
($a_n = 0, \ldots, a_0 = 0$)
⋮
(g) True. #

Example 5. Let S be a set $\{(a_1, a_2) : \ a_1, a_2 \in \mathbb{R}\}$. We define the addition and scaling by

$$(a_1, a_2) + (b_1, b_2) \overset{\text{(define)}}{=} (a_1 + b_1, a_2 - b_2),$$

and

$$c(a_1, a_2) \overset{\text{(define)}}{=} (ca_1, ca_2).$$

Is S a vector space? (NO)

(1) $(a_1, a_2), (b_1, b_2) \in S$. $(a_1, a_2) + (b_1, b_2) = (\underbrace{a_1 + b_1, a_2 - b_2}_{\in S})$

(2) $c(a_1, a_2) = (ca_1, ca_2) \in S$.

(a) $(1,2) + (3,4) = (4,-2)$ $\qquad$ $\vec{a}, \vec{b} \in S$

$(3,4) + (1,2) = (4, 2)$ # $\qquad$ $\vec{a} + \vec{b} \neq \vec{b} + \vec{a}$

Example 6. We consider the set

$$S = \{f \in \mathcal{F}([a, b]) : \ f(a) = 1\}$$

and we define add two functions in S and multiply function in S by a scalar as in Example 3. Is S a vector space? (NO)

(1) $f, g \in S$, $f(a) = 1$, $g(a) = 1$. $(f+g)(a) = f(a) + g(b)$

$= 2$.

$\underline{f+g \notin S}$.

$S = \{ f \in \mathcal{F}([a,b]) \mid f(a) = 0\}$, is a vector space. #

(1) $(f+g)(a) = f(a) + f(a) = 0$, $f+g \in S$. (a)~(g) hold.

(2) $cf(a) = 0$. $cf \in S$.

Example 7. Consider the set S of polynomials of degree equal to n with same addition and scalar multiplication as in Example 4. Then S is not a vector space.

$n \geq 1$ $S = \{ a_n x^n + \cdots + a_0 \text{ with } a_n \neq 0 \}$.

$0 \notin S$.

(1) $\underset{\in S}{x^n} + \underset{\in S}{(-x^n)} = \underset{\notin S}{0}$; (2) $c = 0$

$c x^n = 0 \notin S$.

$n = 4$

$x^2 + 1 \notin S$.

$x^3 - 10^{10000} \notin S$

$0 \notin S$