Today we will discuss

- how to solve a linear system.

- Lecture will be recorded -
Lecture 1: Chapter 1. Linear Algebraic Systems

1.1 The Solution of Linear Systems

What is linear system? For example:

\[
\begin{align*}
5x + 7y + 3z &= 2 \\
2x + y + 6z &= -1 \\
x - 10y + 3z &= 5
\end{align*}
\]

(3 equations with 3 unknowns \(x, y, z\), or

\[
\begin{align*}
w + 5x + 7y + 3z &= 2 \\
2w + 2x + y - 6z &= 1 \\
3w + x + 10y + 3z &= 5 \\
2w - 9x + 4y + 0.22z &= 7
\end{align*}
\]

(4 equations with 4 unknowns \(w, x, y, z\)).

**Goal:** Given such a system of equations, we want to find the variables \(x, y, z, \ldots\) that satisfy all equations simultaneously.

We will learn **Gaussian Elimination**, that is to reduce the original system to a much **simpler system** that still has the **same solution**.
Example 1: Find solutions of the following linear system:

\[
\begin{align*}
    x - 2y + z &= 3 \\
    2x - y - 2z &= 6 \\
    3x - 7y + 4z &= 10
\end{align*}
\]

1. Fix \( x \) in \( 0 \). Use \( 0 \) to eliminate \( "x" \) in \( 2 \) \( 3 \).

\[
\begin{align*}
    2 \cdot 2 - 2 \cdot 1 : & \quad 2x - y - 2z = 6 \\
    -) 2x - 4y + z &= 6. \\
    \hline
    y - 4z &= 0. & \quad \text{New 2}
\end{align*}
\]

\[
\begin{align*}
    3 \cdot 3 - 3 \cdot 1 : & \quad 3x - 7y + 4z = 10 \\
    -) 3x - 6y + 3z &= 9 \\
    \hline
    -y + z &= 1. & \quad \text{New 3}
\end{align*}
\]

New system:

\[
\begin{align*}
    x - 2y + z &= 3 \\
    3y - 4z &= 0 \\
    -y + z &= 1
\end{align*}
\]

has \( "\text{the same solution as the original one, but is simpler}" \)

2. Fix \( y \) in \( 3 \), Use \( 3 \) to eliminate \( "y" \) in \( 3 \).

\[
\begin{align*}
    3 + \frac{1}{3} 2 : & \quad -4z + 2z = 1 \Rightarrow \frac{-2}{3} z = 1.
\end{align*}
\]

3. Solve \( x \), \( y \), \( z \).
Remark: (1) If we have $n$ equations, $n$ knowns:

\[
\begin{align*}
\text{equ.}(1) & \quad \text{Use (1) to eliminate "1st variable" from (2) \ldots (n)} \\
\text{equ.}(2) & \quad \text{Use (2) "2nd \ldots" from (3) \ldots (n)} \\
\vdots & \quad \vdots \\
\text{equ.}(n) & \\
\end{align*}
\]

Finally we get "upper triangular form".

This is called "Gaussian elimination".

(2) **Back substitution:** Solve this triangular from system from bottom up.
1.2 Matrices and Vectors and Basic Operations

A **matrix** is simply a rectangle array of numbers, such as,

\[
\begin{pmatrix}
1 & 0.7 & 10 & 0 \\
\pi & 6 & 0 & -2
\end{pmatrix}, \quad \begin{pmatrix}
\cos(1) & 1 \\
4 & 6 \\
-10 & e^2
\end{pmatrix}
\]

The 1st matrix above is a $2 \times 4$ matrix and 2nd matrix above is a $3 \times 2$ matrix.

Generally, an $m \times n$ matrix $A$ is a two-dimensional array of $m \cdot n$ numbers:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

where $m$ is the number of rows and $n$ is the number of columns. The element $a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, is called the **entry** of $A$.

A **column vector** is a matrix where $n = 1$:

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}
\]

A **row vector** is a matrix where $m = 1$:

\[
w = (w_1 \ w_2 \ \cdots \ \ w_n)
\]
§ Three basic operations:

1. **Matrix addition:**

   \[
   \begin{pmatrix}
   a_{11} & a_{12} & \cdots & a_{1n} \\
   a_{21} & a_{22} & \cdots & a_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{m1} & a_{m2} & \cdots & a_{mn}
   \end{pmatrix}
   \]

   \[
   \begin{pmatrix}
   b_{11} & b_{12} & \cdots & b_{1n} \\
   b_{21} & b_{22} & \cdots & b_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   b_{m1} & b_{m2} & \cdots & b_{mn}
   \end{pmatrix}
   \]

   \[
   = \begin{pmatrix}
   a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\
   a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
   \end{pmatrix}
   \]

2. **Scalar multiplication:** If \( c \) is a number, we can multiply a matrix by \( c \):

   \[
   c \times \begin{pmatrix}
   a_{11} & a_{12} & \cdots & a_{1n} \\
   a_{21} & a_{22} & \cdots & a_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{m1} & a_{m2} & \cdots & a_{mn}
   \end{pmatrix}
   = \begin{pmatrix}
   c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\
   c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   c \cdot a_{m1} & c \cdot a_{m2} & \cdots & c \cdot a_{mn}
   \end{pmatrix}
   \]
3. Matrix multiplication:

\[
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_p
\end{pmatrix}
\times
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_p
\end{pmatrix}
= v_1 w_1 + v_2 w_2 + \cdots + v_p w_p
\]

Generally, if \( A = (a_{ij}) \) is an \( m \times n \) matrix and \( B = (b_{ij}) \) is an \( n \times p \) matrix, then their product \( C = AB \) is an \( m \times p \) matrix and has entries:

\[
c_{ij} = (i^{\text{th}} \text{ row of } A) \times (j^{\text{th}} \text{ column of } B)
\]

\[
C = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1p} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{m1} & b_{m2} & \cdots & b_{mp}
\end{pmatrix}
= \begin{pmatrix}
  c_{11} & c_{12} & \cdots & c_{1p} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{m1} & c_{m2} & \cdots & c_{mp}
\end{pmatrix}
\]

where

\[
c_{11} = (a_{11}, a_{12}, \ldots, a_{1n}) \cdot (b_{11}, \ldots, b_{1p}) = a_{11} b_{11} + \cdots + a_{1n} b_{1p}
\]

\[
c_{12} = (a_{11}, \ldots, a_{1n}) \cdot (b_{12}, \ldots, b_{1p}) = a_{11} b_{12} + \cdots + a_{1n} b_{1p}
\]

\[
\vdots
\]

Remark:

- Matrix multiplication is associative: \((AB)C = A(BC)\)
- Not commutative: in general, \(AB \neq BA\).
Example 1: Let $A = (0, 1, 2)$ and 

$$B = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 1 & 5 \end{pmatrix}.$$ 

Compute $AB$ and $3B$. Can we compute $BA$?

$$AB = \begin{pmatrix} 0 & 1 \\ 2 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 14 \end{pmatrix}.$$ 

$$3B = \begin{pmatrix} 3 & 9 \\ 0 & 12 \\ 3 & 15 \end{pmatrix}.$$

 Scalars, matrices give a convenient notation for linear systems. For example,

linear system

\begin{align*}
    x - 2y + z &= 3 \\
    2x - y - 2z &= 6 \\
    3x - 7y + 4z &= 10
\end{align*}

is equivalent to:

$$\begin{pmatrix}
    1 & -2 & 1 \\
    2 & -1 & -2 \\
    3 & -7 & 4
\end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}.$$ 

In more compact notation, we can write: $A \mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -2 \\ 3 & -7 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}.$$ 

* Note that the solution $\mathbf{x}$ was obtained in Example 1 above.
Some special matrices and notations that we will see and utilize many times in this course.

- The \( n \times n \) identity matrix, typically denoted \( I \) or \( I_n \), defined by:

\[
I = I_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}_{n \times n}
\]

In other words, \( I \) has 1’s on the main diagonal, and the off-diagonal elements are 0. It’s easy to check that

\[
I_n A = A \quad \text{and} \quad B I_n = B
\]

for any matrix \( A \) with \( n \) rows and any matrix \( B \) with \( n \) columns.

- The \( m \times n \) zero matrix, typically denoted \( O \) or \( O_{m \times n} \), which has all zero entries. It’s easy to check that

\[
O_{m \times n} A = O_{m \times k} \quad \text{for any } n \times k \text{ matrix } A
\]

and

\[
B O_{m \times n} = O_{k \times n} \quad \text{for any } k \times m \text{ matrix } B.
\]

- We denote by \( \text{diag}(a_1, \ldots, a_n) \) the following \( n \times n \) matrix:

\[
\text{diag}(a_1, \ldots, a_n) = \begin{pmatrix}
a_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & a_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & a_n
\end{pmatrix}
\]

Using this notation, \( I_n = \text{diag}(1, \ldots, 1) \)
The augmented matrix for a linear system appends the right hand side as an extra column to the coefficient matrix.

For example, the augmented matrix for the linear system

\[
\begin{align*}
    x + 2y + 2z &= 2 \\
    2x + 6y &= 1 \\
    4x + 4z &= 0
\end{align*}
\]

is the 3-by-4 matrix:

\[
\begin{pmatrix}
    1 & 2 & 2 & 2 \\
    2 & 6 & 0 & 1 \\
    4 & 0 & 4 & 0
\end{pmatrix}
\]

For clarity, the augmented matrix can also be written as:

\[
\begin{pmatrix}
    1 & 2 & 2 & 2 \\
    2 & 6 & 0 & 1 \\
    4 & 0 & 4 & 0
\end{pmatrix}
\]

Gaussian elimination can be expressed entirely in terms of the augmented matrix. Also, the operations of Gaussian elimination can be used to update the augmented matrix.