Lecture 1: Welcome to M4242

Today we will discuss

- how to solve a **linear system**.

- Lecture will be recorded -

Lecture 1: Chapter 1. Linear Algebraic Systems
1.1 The Solution of Linear Systems

What is linear system? For example:

$$
\begin{aligned}
5x + 7y + 3z &= 2 \\
2x + y + 6z &= -1 \\
x - 10y + 3z &= 5
\end{aligned}
$$

linear system.

(3 equations with 3 unknowns x, y, z), or

$$
\begin{aligned}
w + 5x + 7y + 3z &= 2 \\
2w + 2x + y - 6z &= 1 \\
3w + x + 10y + 3z &= 5 \\
2w - 9x + 4y + 0.22z &= 7
\end{aligned}
$$

(4 equations with 4 unknowns w, x, y, z).

Goal: Given such a system of equations, we want to find the variables $x, y, z, \ldots$ that satisfy all equations simultaneously.

We will learn **Gaussian Elimination**, that is to reduce the original system to a much simpler system that still has the same solution.

Example 1: Find solutions of the following linear system:

$$x - 2y + z = 3 \quad - ①$$
$$2x - y - 2z = 6 \quad - ②$$
$$3x - 7y + 4z = 10 \quad - ③$$

1. Fix x in ①. Use ① to eliminate "x" in ② ③.

② − 2①:
$$\begin{array}{rl} & 2x - y - 2z = 6 \\ -) & 2x - 4y + 2z = 6. \\ \hline & 3y - 4z = 0. \end{array} \quad - \text{New } ②$$

③ − 3①:
$$\begin{array}{rl} & 3x - 7y + 4z = 10 \\ -) & 3x - 6y + 3z = 9 \\ \hline & -y + z = 1 \end{array} \quad - \text{New } ③$$

New system:
$$\begin{cases} x - 2y + z = 3 \\ 3y - 4z = 0 \\ -y + z = 1 \end{cases}$$

has "the same" solution as the original one, but is simpler.

2. Fix y in ②, Use ② to eliminate "y" in ③.

③ + $\frac{1}{3}$ ② $\longrightarrow$ $-\frac{4}{3}z + z = 1 \Rightarrow -\frac{1}{3}z = 1.$

$$\begin{cases} x - 2y + z = 3 & - ① \\ 3y - 4z = 0 & - ② \\ -\frac{1}{3}z = 1 & - ③ \end{cases}$$

triangular form.

3. Solve x, y, z.

[Example Continue]

$③$: $-\frac{1}{3}z = 1 \Rightarrow \underline{z = -3}$

$②$: $3y - 4(-3) = 0 \Rightarrow \underline{y = -4}$

$①$: $x - 2(-4) + (-3) = 3 \Rightarrow \underline{x = -2}$

$(x, y, z) = (-2, -4, -3).$

Remark: (1) If we have n equations, n knowns:

$$\begin{cases} equ.(1) \\ equ.(2) \\ \vdots \\ equ.(n) \end{cases}$$

Use (1) to eliminate "1st variable" from (2) ... (n)

Use (2) " "2nd " " from (3) .... (n)

⋮

Finally we get "upper triangular form".

This is called "Gaussian elimination".

(2) **Back substitution:** Solve this triangular from system from bottom up.

1.2 Matrices and Vectors and Basic Operations

A **matrix** is simply a rectangle array of numbers, such as,

1st column

$$\begin{array}{l} \text{1st row} \\ \text{2nd row} \end{array} \begin{pmatrix} 1 & 0.7 & 10 & 0 \\ \pi & 6 & 0 & -2 \end{pmatrix}, \qquad \begin{pmatrix} \cos(1) & 1 \\ 4 & 6 \\ -10 & e^2 \end{pmatrix}$$

The 1st matrix above is a $\underbrace{2 \times 4}_{\text{size}}$ matrix and 2nd matrix above is a $\underbrace{3 \times 2}_{\text{size.}}$ matrix.

Generally, an $m \times n$ matrix A is a two-dimensional array of $m \cdot n$ numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

a_{12} : 1st row, 2nd column.

where m is the number of rows and n is the number of columns.
The element a_{ij}, $1 \le i \le m$, $1 \le j \le n$, is called the **entry** of A.

A **column vector** is a matrix where $n = 1$:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

A **row vector** is a matrix where $m = 1$:

$$\mathbf{w} = (w_1 \, w_2 \, \cdots \, w_n)$$

§ Three basic operations:

same size.

1. **Matrix addition:**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

2. **Scalar multiplication:** If c is a number, we can multiply a matrix by c:

$$c \times \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \cdots & c \cdot a_{mn} \end{pmatrix}$$

3. **Matrix multiplication:**

like "dot product"

$(v_1, \dots, v_p) \cdot (w_1, \dots, w_p) = v_1 w_1 + \cdots + v_p w_p$

column vector

$$\underset{\text{row vector}}{\begin{pmatrix} v_1 & \cdots & v_p \end{pmatrix}}_{1\times p} \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}_{p\times 1} = v_1 w_1 + v_2 w_2 + \cdots + v_p w_p$$

Generally, if $A = (a_{ij})$ is $m \times n$ matrix and $B = (b_{ij})$ is $n \times p$ matrix, then their product $C = AB$ is $m \times p$ matrix and has entries:

$$c_{ij} = (i^{\text{th}} \text{ row of } A) \times (j^{\text{th}} \text{column of } B)$$

* number of columns of A = number of rows of B.

$$C = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ \vdots & \vdots & & \\ c_{m1} & \cdots & \cdots & c_{mp} \end{pmatrix}_{m\times p}$$

where $c_{11} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix} = a_{11}b_{11} + \cdots + a_{1n}b_{n1}$

$c_{12} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \end{pmatrix} \begin{pmatrix} b_{12} \\ \vdots \\ b_{n2} \end{pmatrix} = a_{11}b_{12} + \cdots + a_{1n}b_{n2}$

$\vdots$

Remark:

- Matrix multiplication is associative: $(AB)C = A(BC)$
- Not commutative: in general, $AB \neq BA$.

Example 1: Let $A = (0, 1, 2)$ and 1×3

$$B = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 1 & 5 \end{pmatrix}_{3\times 2}.$$

Compute AB and $3B$. Can we compute BA? (NO).

$$AB = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 14 \end{pmatrix}_{1\times 2}$$

$$3B = \begin{pmatrix} 3 & 9 \\ 0 & 12 \\ 3 & 15 \end{pmatrix}_{3\times 2}.$$

§ **Vectors, matrices give a convenient notation for linear systems.**
For example,

linear system

$$\begin{cases} x - 2y + z &= 3 \\ 2x - y - 2z &= 6 \\ 3x - 7y + 4z &= 10 \end{cases} \quad \text{is equivalent to :} \quad \underbrace{\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -2 \\ 3 & -7 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}}_{\vec{b}}$$

In more compact notation, we can write: $\boxed{A\,\mathbf{x} = \mathbf{b}}$, where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -2 \\ 3 & -7 & 4 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}$$

* Note that the solution $\mathbf{x}$ was obtained in Example 1 above.

§ **Some special matrices and notations** that we will see and utilize many times in this course.

- The n-by-n **identity matrix**, typically denoted I or I_n, defined by:

$$I = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n\times n}$$

  In other words, I has 1's on the main diagonal, and the off-diagonal elements are 0. It's easy to check that

$$I_n A = A \quad \text{and} \quad B I_n = B$$

  for any matrix A with n rows and any matrix B with n columns.

- The m-by-n **zero matrix**, typically denoted O or $O_{m\times n}$, which has all zero entries. It's easy to check that

$$O_{m\times n} \underset{n\times k}{A} = O_{m\times k} \qquad \text{for any } n\text{-by-}k \text{ matrix } A$$

  and

$$\underset{k\times m}{B} O_{m\times n} = O_{k\times n} \qquad \text{for any } k\text{-by-}m \text{ matrix } B.$$

- We denote by $\operatorname{diag}(a_1, \ldots, a_n)$ the following n-by-n matrix:

$$\operatorname{diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

  Using this notation, $I_n = \operatorname{diag}(\underbrace{1, \ldots, 1}_{n \text{ times}})$

§ The **augmented matrix** for a linear system appends the right hand side as an extra column to the coefficient matrix.

For example, the augmented matrix for the linear system

$$\begin{aligned} x + 2y + 2z &= 2 \\ 2x + 6y &= 1 \\ 4x + 4z &= 0 \end{aligned}$$

is the 3-by-4 matrix:

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

For clarity, the **augmented matrix** can also be written as:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 0 \end{array}\right)$$

Gaussian elimination can be expressed entirely in terms of the augmented matrix. Also, the operations of **Gaussian elimination** can be used to update the augmented matrix.