# Lecture 10: Quick review from previous lecture

- **Definition:** A **vector space** is a set $V$ equipped with two operations:

  (1) (Addition) If $\mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} + \mathbf{w} \in V$.

  (2) (Scalar Multiplication) Multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.

  For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

  (a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

  (b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

  (c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.

  (d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.

  (e) *Distributivity*: $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.

  (f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.

  (g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.

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Today we will discuss

- Sec. 2.2 Subspace and Sec. 2.3 Span and Linear Independence.

**- Lecture will be recorded -**

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## 2.2 Subspaces

vector space

**Definition:** If $W \subset \boxed{V}$ (that is, $W$ is a "subset" of $V$) and $W$ is a vector space under the same addition and scalar multiplication defined on a vector space $V$, then $W$ is called a **subspace** of $V$.

→ ① $W \subset V$
② $W$ is vector space.

✓ "Subspaces" are vector spaces that are embedded in larger vector spaces.

If we want to check if $W \subset V$ is a subspace of $V$, it is enough to check the following 3 conditions:

1. $W$ must contain **zero element** of $V$
2. If $\mathbf{v}$ and $\mathbf{w}$ in $W$, then $\mathbf{v} + \mathbf{w} \in W$.
3. If $\mathbf{v} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in W$.

**Example 1:**

(1) $W = \{0\}$ is the trivial subspace of the vector space $\mathbb{R}^n$.

① $0 \in W$

② $0 + 0 = 0 \in W$

③ $c\,0 = 0 \in W$

(2) $S = \{(x, y, 0)^T\}$ is a subspace of the vector space $\mathbb{R}^3$.

① $(0,0,0)^T \in S$.

② $(x, y, 0)^T + (a, b, 0)^T = (x+a, y+b, 0)^T \in S$.

③ $c(x, y, 0)^T = (cx, cy, 0)^T \in S$.

**Example 2:**

(1) $S = \{(x, y, 1)^T\}$ is NOT a subspace of the vector space $\mathbb{R}^3$.

① $(0,0,0) \notin S$.

② $(a,b,1) + (c,d,1) = (a+c, b+d, 2) \notin S$.

③ $7(1,1,1) = (7,7,7) \notin S$.

(2) Is $S = \{x \geq 0,\ y \geq 0,\ z = 0\}$ a subspaces of $\mathbb{R}^3$?

① $(0,0,0) \in S$.

② ok

③ $-7(1,1,0) = (-7,-7,0) \notin S$.

(3) Another interesting example is the space of solutions to a linear homogeneous differential equation on $[a, b]$, for example,

homogeneous

$$S = \{u \in \mathcal{F}([a, b]) :\ u \text{ is the solution to } u''(x) + 9u(x) = \boxed{0}\}.$$

(Yes) Is $S$ a subspace of $\mathcal{F}([a, b])$? Recall $\mathcal{F}([a, b])$ is the collection of all functions $f$ defined on an interval $[a, b]$

vector space.

① $0'' + 9\,0 = 0$. $0 \in S$.

② $u, w \in S$. $\begin{cases} u'' + 9u = 0 \\ w'' + 9w = 0. \end{cases}$

Check $u+w \in S$: $(u+w)'' + 9(u+w) = u'' + w'' + 9u + 9w$

$= 0 + 0 = 0$.

Then $u+w \in S$.

③ $u \in S$, Check $cu \in S$: $(cu)'' + 9(cu) = c(u'' + 9u)$

$= 0$.

**Remark:** 0 is essential for Example 2 (3) above.

$S = \{ u \mid u'' + 9u = 17 \}$

Is NOT subspace

Recall: We denote the set of all $m \times n$ matrices with entries from $\mathbb{R}$ by

$\mathsf{M}_{m\times n}(\mathbb{R}) := \{A : A \text{ is } m \times n \text{ matrix}\}$ (Recall that it is a vector space)

**Example 3:** The set of all $3 \times 3$ **upper triangular matrix** is a subspace of $\mathsf{M}_{3\times 3}(\mathbb{R})$.

$$S = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} : a, b, \dots, f \in \mathbb{R} \right\}. \subseteq M_{3\times 3}.$$

① $O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S.$

② $A, B \in S.$ $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, B = \begin{bmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ 0 & \tilde{d} & \tilde{e} \\ 0 & 0 & \tilde{f} \end{bmatrix}$

$$A + B = \begin{bmatrix} a+\tilde{a} & b+\tilde{b} & c+\tilde{c} \\ 0 & d+\tilde{d} & e+\tilde{e} \\ 0 & 0 & f+\tilde{f} \end{bmatrix} \in S.$$

③ $cA \in S.$

## 2.3 Span and Linear Independence

**Definition:** Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space $V$. If we take any scalars $c_1, \ldots, c_n$, we can form a new vector in $V$ as follows:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

linear combination

An expression of this kind is known as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

**Example 1.** If we have vectors $\mathbf{v}_1 = (1,2)^T$, $\mathbf{v}_2 = (-1,0)^T$ and $\mathbf{v}_3 = (2,-1)^T$ in $\mathbb{R}^2$, we can form the linear combination

$$2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = 2(1,2)^T - (-1,0)^T + 3(2,-1)^T = \boxed{(9,1)^T}$$

l. combination of $v_1, v_2, v_3$.

(zero element)

**Example 2.** We observe that $0\mathbf{v} = \mathbf{0}$ for each $\mathbf{v} \in V$. Thus $\mathbf{0}$ vector is a linear combination of any nonempty subset of $V$.

**Definition:** If we fix some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space $V$, we can consider the set of **all** of their linear combinations, This set is called the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, denoted

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

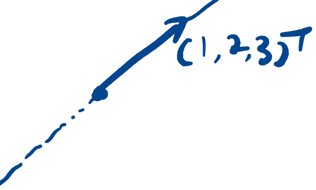
In other words,

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \left\{ \sum_{i=1}^{n} c_i\mathbf{v}_i : c_1, \ldots, c_n \in \mathbb{R} \right\}$$

**Remark:** In fact, $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of $V$.

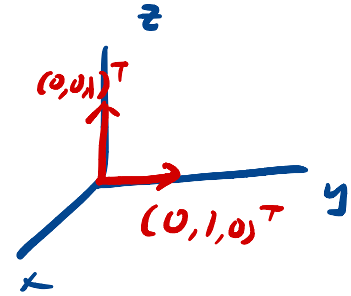
1. $0v_1 = \mathbf{0} \in \text{span}\{v_1, \ldots, v_n\}$.
2. $\sum a_i v_i + \sum b_i v_i = \sum (a_i + b_i) v_i \in \text{span}\{v_1, \ldots, v_n\}$.
3. $c(\sum a_i v_i) = \sum (ca_i) v_i \in \text{span}\{v_1, \ldots, v_n\}$.

**Example 3.** (1) Let $\mathbf{v}_1 = (1,2,3)^T$. What does $\text{span}\{\mathbf{v}_1\}$ consist of in $\mathbb{R}^3$?




$\text{span}\{v_1\} = \{ c v_1 \mid c \in \mathbb{R} \}$. line.

(2) What does $\text{span}\{(0,1,0)^T, (0,0,1)^T\}$ consist of in $\mathbb{R}^3$?



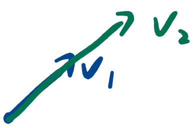

$\text{span}\{(0,1,0)^T, (0,0,1)^T\}$

$= \{ a(0,1,0)^T + b(0,0,1)^T \mid a, b \in \mathbb{R} \}$.

$= \{ (0,a,b)^T \mid a, b \in \mathbb{R} \}$ = $yz$-plane

**Example 4.** If $\mathbf{v}_1 = c\mathbf{v}_2$ in $\mathbb{R}^3$, then what is $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$? $= \text{span}\{v_1\} = \text{span}\{v_2\}$.

(parallel)




$$\text{span}\{v_1, v_2\} = \{ a v_1 + b v_2 \mid a, b \in \mathbb{R} \}$$
$$= \{ a(cv_2) + b v_2 \mid a, b \in \mathbb{R} \}.$$
$$= \text{span}\{v_2\}.$$

**Remark:**

- If $\mathbf{v}_1 \neq 0$ in $\mathbb{R}^3$, then $\text{span}\{\mathbf{v}_1\}$ is the line $\{c\mathbf{v}_1 : c \in \mathbb{R}\}$.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-zero vectors in $\mathbb{R}^3$ that are not parallel to each other (i.e. $\mathbf{v}_1 \neq c\mathbf{v}_2$ for any scalar $c$), then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ defines a *plane*.

**Example 5.** Determine the span of $f_1(x) = 1,\ f_2(x) = x,\ f_3(x) = x^2$.

$$\text{span}\{1, x, x^2\} = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}.$$
$$= P^{(2)}$$

$P^{(n)}$ (poly. of degree $\le n$) $= \text{span}\{1, x, \dots, x^n\}$.

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}$$

**Example 6.** The span of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $M_{2\times 2}(\mathbb{R})$.

$$\text{span}\{A_1, \ldots, A_4\} = \{aA_1 + bA_2 + cA_3 + dA_4 \mid a,b,c,d \in \mathbb{R}\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

$$= M_{2\times 2}(\mathbb{R}).$$

Note that $\text{span}\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\} = \mathbb{R}^3$.

**Example 7.** Let $\mathbf{v}_1 = (1,0,0)^T$, $\mathbf{v}_2 = (0,1,1)^T$, $\mathbf{v}_3 = (1,0,1)^T$. Show $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$. ① $\text{span}\{v_1, v_2, v_3\} \subseteq \mathbb{R}^3$.

② $\text{span}\{v_1, v_2, v_3\} \supseteq \mathbb{R}^3$

Take any vector $(x, y, z)^T \in \mathbb{R}^3$, to check if $(x,y,z)^T$ is l. combination of $v_1, v_2, v_3$.

If so, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a v_1 + b v_2 + c v_3$,

Find $a, b, c$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

If $A$ is invertible, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\underline{\det A = 1} \neq 0$. So $A$ is invertible.

This implies $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ really exists, and thus $(x,y,z)^T$ is a l. combination of $v_1, v_2, v_3$.

**Poll Question 1:** With the usual matrix addition and scalar multiplication

$$\mathsf{M}_{4\times 5}(\mathbb{R}) := \{A : \ A \text{ is } 4\times 5 \text{ matrix}\}$$

is a vector space.

$A$) Yes
$B$) No

**Poll Question 2:** With the usual matrix addition and scalar multiplication

$$\mathsf{M}_{2\times 2}(\mathbb{R}) := \{A : \ A \text{ is } 2\times 2 \text{ matrix with the form } A = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}\}$$

is a vector space.

$A$) Yes
$B$) No

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.

Caution: after clicking submit, you will not be able to resubmit your answer!