Lecture 11: Quick review from previous lecture

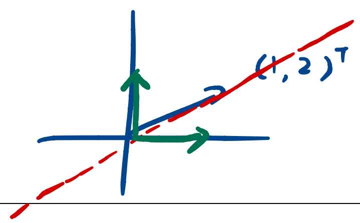
- To check if $W \subset V$ is a subspace of V, it is enough to check the following 3 conditions:
  1. W must contain zero element of V,
  2. If $\mathbf{v}$ and $\mathbf{w}$ in W, then $\mathbf{v} + \mathbf{w} \in W$,
  3. If $\mathbf{v} \in W$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in W$.
- A linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

- We define the set of collecting all possible linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \left\{ \sum_{i=1}^{n} c_i\mathbf{v}_i : c_1, \ldots, c_n \in \mathbb{R} \right\}$$

EX:



$\text{span}\{(1,2)^T\} = \{c(1,2)^T \mid c \in \mathbb{R}\}$.

$\text{span}\{(1,0)^T, (0,1)^T\} = \mathbb{R}^2$

Today we will discuss

- the linear independent (dependent).

- Lecture will be recorded -

HW is due Today 6pm.

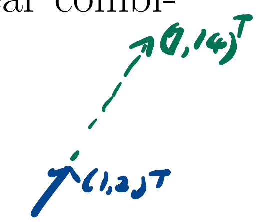
§ Linear Independence and Dependence

Definition: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space V, we say they are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all of which are zero, so that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are not linearly dependent, we say they are **linearly independent**.

In other words, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent** if the *only* linear combination $\sum_{i=1}^{n} c_i\mathbf{v}_i$ that is equal to $\mathbf{0}$ is when all the c_i's are equal to 0.



Example 8.

(1) $(7, 14)^T = 7(1, 2)^T$. Thus $(1, 2)^T, (7, 14)^T$ are linearly dependent.

$$1\underbrace{(7,14)^T}_{v_1} - 7\underbrace{(1,2)^T}_{v_2} = (0,0)^T$$

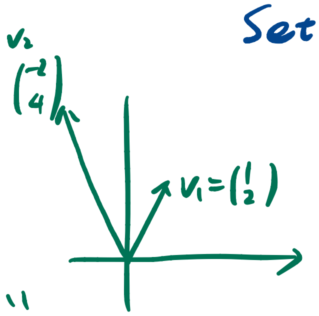
$c_1 = 1$, $c_2 = -7$.

$$\overbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}^{A}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(2) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ are linearly independent. Why?

Set up $a v_1 + b v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Find a, b.

$$a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



"$[v_1, v_2]$ are not parallel"

$$\begin{cases} a - 2b = 0 \\ 2a + 4b = 0 \end{cases}$$

homogeneous l. system.

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{②-2①} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 8 & 0 \end{array}\right)$$

Then ② $8b = 0$. $b = 0$

① $a - 2b = 0$, $a = 0$.

So $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. By definition, $\{v_1, v_2\}$ l. indep.

Example 9. Determine if $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$ are linearly independent or not?

Set $a v_1 + b v_2 + c v_3 = 0$. Find a, b, c.

$$\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \xrightarrow{②+①} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right].$$

$$\xrightarrow{③-②} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Q: rank $A = 2$:

Free variable : c.

② $3b + 6c = 0 \Rightarrow b = -2c$.

① $a = -c$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} c, \quad \forall c \in \mathbb{R}.$$

We can then conclude that We have $\{v_1, v_2, v_3\}$ are l. dep since we can choose $c \neq 0$.

Fact 1: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ and let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$:

(1) $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly **dependent** if and only if there is a nonzero solution $\mathbf{x}$ to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. (Ex 9)

(2) $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly **independent** if and only if the only solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is the trivial one, $\mathbf{x} = \mathbf{0}$. (Ex 8, (2))

(3) A vector $\mathbf{b} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ if and only if $A\mathbf{x} = \mathbf{b}$ is compatible (i.e., has at least one solution). $b = c_1 v_1 + \cdots + c_k v_k$. (has solution(s))

$$A x = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = c_1 v_1 + \cdots + c_k v_k$$

Q: Suppose we take any four vectors in $\mathbb{R}^3$; call them $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$. Can they be linearly independent?

Example 10. For instance, we take the 4 vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}.$$

Setup $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Find $x_1, \dots, x_4$

$A = [v_1 \ v_2 \ v_3 \ v_4]$. $\boxed{A \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 0 & -4 & 2 \\ -1 & 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A \xrightarrow[③+①]{②-2①} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 7 & 7 & 7 \end{bmatrix} \xrightarrow{③+\frac{7}{6}②} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q: rank A : 2

free variables: x_3, x_4

② $6x_2 + 6x_3 + 6x_4 = 0$, $\quad x_2 = -x_3 - x_4$.

① $x_1 + 3x_2 + x_3 + 4x_4 = 0$.

$x_1 = -3(-x_3 - x_4) - x_3 - 4x_4$.

$= 2x_3 - x_4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_3 - x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$

Then $\{v_1, \dots, v_4\}$ are l. dep.

We can use the same logic to show the general fact: if $\mathbf{v}_1, \dots, \mathbf{v}_k$ are k vectors in $\mathbb{R}^n$, then they must be linearly dependent if $k > n$.

In other words, we have the following fact:

Fact 2: If $k > n$, then any set of k vectors in $\mathbb{R}^n$ is linearly dependent.

Example 11. Determine whether

$$\{x^3 + 2x^2,\ -x^2 + 3x + 1,\ x^3 - x^2 + 2x - 1\}$$

in $\mathcal{P}^{(3)}$ are linearly independent or linearly dependent.

$a(x^3 + 2x^2) + b(-x^2 + 3x + 1) + c(x^3 - x^2 + 2x - 1) = 0$

Find a, b, c.

$$(\underbrace{a+c}_{=0})x^3 + (\underbrace{2a-b-c}_{=0})x^2 + (\underbrace{3b+2c}_{=0})x + (\underbrace{b-c}_{=0}) = 0$$

$$\begin{cases} a + c = 0 \\ 2a - b - c = 0 \\ 3b + 2c = 0 \\ b - c = 0 \end{cases} \qquad \overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}}^{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

, $\underline{\text{rank } A = 3}$

$$\Longrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then they are l. <u>indep.</u> #

n+1 polynomials

Remark: It can be shown that $\{1, x, \ldots, x^n\}$ are linearly independent in $\mathcal{P}^{(n)}$.

Fact 2: Let $k \leq n$. A set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in $\mathbb{R}^n$ is **linearly independent** if and only if the rank of $A = [\mathbf{v}_1, \dots, \mathbf{v}_k]$ is equal to k.

To be continued!

Fact 3: If $\mathbf{v}_n$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$, then

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n\} = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}.$$

*See also Example 4: If $\mathbf{v}_1 = c\mathbf{v}_2$, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\}$.

Example 12. Determine whether

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

in $\underline{M_{2\times 2}(\mathbb{R})}$ are linearly independent or linearly dependent.

$aA + bB + cC = O_{2\times 2}$. Find a, b, c.

$$a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{cases} a + b + c = 0 \\ a + 2c = 0 \\ b + c = 0 \\ a + b + c = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$