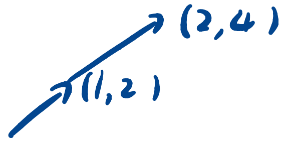
Lecture 12: Quick review from previous lecture

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in a vector space V, we say they are **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all of which are zero, so that

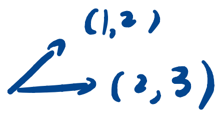
$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

If all c_i can only be zero, then we call $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent**.

Ex:



, $\{(1,2), (2,4)\}$ l. dep.



$\{(1,2), (2,3)\}$ l. indep.

Today we will discuss

- Sec. 2.4 Basis and Dimension.

- Lecture will be recorded -

Fact 2: Let $k \leq n$. A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is **linearly independent** if and only if the rank of $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$ is equal to k.

($\Leftarrow$) Set up $c_1 v_1 + \cdots + c_k v_k = 0$. Show $c_1 = \cdots = c_k = 0$.

Since $\operatorname{rank}(A) = k$, we have

$$A \xrightarrow[\text{ops.}]{\text{row}} U = \begin{bmatrix} a_1 & \cdots & \ast \\ & \ddots & \\ 0 & & a_k \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n\times k} = \begin{bmatrix} B \\ \hline O \end{bmatrix}_{n\times k}$$

$a_1 \neq 0$, $\vdots$, $a_k \neq 0$.

where $B = \begin{bmatrix} a_1 & & \ast \\ & \ddots & \\ 0 & & a_k \end{bmatrix}$ is indeed nonsingular.

$$c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow A \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow B \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = \cdots = c_k = 0 \text{ since } B \text{ is nonsingular}$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = B^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \#$$

($\Rightarrow$) skip.

Fact 3: If $\mathbf{v}_n$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, then

$$\underbrace{\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n\}}_{S_n} = \underbrace{\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}}_{S_{n-1}}.$$

*See also Example 4: If $\mathbf{v}_1 = c\mathbf{v}_2$, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \operatorname{span}\{\mathbf{v}_1\}$.

It's clear to see $S_n \supseteq S_n$. ("$A \supseteq B$": A contains B)

We only need to show $S_n \subseteq S_{n-1}$

For any $w \in S_n$, we can express w as a l. combination of $v_1, \ldots, v_n$, that is,

$$w = c_1 v_1 + \cdots + c_n v_n$$

since v_n is l. comb. of $v_1, \ldots, v_{n-1}$

$$= c_1 v_1 + \cdots + c_{n-1} v_{n-1} + c_n (a_1 v_1 + \cdots + a_{n-1} v_{n-1})$$

$$= (c_1 + a_1 c_n) v_1 + \cdots + (c_{n-1} + a_{n-1} c_n) v_{n-1}$$

$$\in \operatorname{span}\{v_1, \ldots, v_{n-1}\} = S_{n-1}.$$

Then $S_n \subseteq S_{n-1}$. #

✓ $\mathbf{v}_n$ is redundant in view of $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$.

2.4 Basis and Dimension

Definition:

(1) If $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, we say that $\mathbf{v}_1, \dots, \mathbf{v}_n$ **span** V.
(generate)

(2) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V and are linearly independent, we say that they form a **basis** of a vector space V.

*So a basis for a vector space V is a linearly independent set of vectors that span V.

Example 1.

(1) Check $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$, $\mathbf{e}_3 = (0,0,1)^T$ are linearly independent.

$a_1 e_1 + a_2 e_2 + a_3 e_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Find a_1, a_2, a_3.

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow a_1 = a_2 = a_3 = 0.$$

(2) We have known that $\text{span}\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\} = \mathbb{R}^3$. Thus, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$.

In general, the "standard basis" of $\mathbb{R}^n$ consists of the n vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Here $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent and they span $\mathbb{R}^n$, since any vector $\mathbf{x} = (x_1, \dots, x_n)^T$ can be written as $\boxed{\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i.}$ #

A natural question is: can there be a basis of $\mathbb{R}^n$ with a different number of vectors (not n)?

The answer is "no"! In fact

Fact 1: Any basis of $\mathbb{R}^n$ must have exactly n vectors.
In addition, a set of $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if and only if $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is nonsingular ($\mathsf{rank}(A) = n$).

[To see this] From Fact 2 in Section 2.3, we have

($\Rightarrow$) $\{v_1, \ldots, v_n\}$ is a basis $\Longrightarrow$ $\{v_1, \ldots, v_n\}$ l. indep.

$\overset{\text{Fact 2, Sec 2.3}}{\Longleftrightarrow}$ $A = [v_1 \cdots v_n]$ has rank $\underline{\underline{n}}$

($\Leftarrow$) Exercise. $\Longleftrightarrow$ A is nonsingular.

Now, we consider the fact

Fact 2: Let $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set of n linearly independent vectors, then any set of k elements $\mathbf{w}_1, \ldots, \mathbf{w}_k$ in V with $k > n$ is linearly dependent.

Then we can show the general case.

Fact 3: If V is any vector space that has a basis with n vectors, then any other basis must also have n vectors.

[To see this] Suppose V has a basis $\{v_1, \ldots, v_n\}$, and also it has another basis $\{w_1, \ldots, w_k\}$.

Show $k = n$.

① V has a basis $\{v_1, \ldots, v_n\}$ ($V = \text{span}\{v_1, \ldots, v_n\}$) then $k \le n$ o/w $\{w_1, \ldots, w_k\}$ are l. dep.

② V has a basis $\{w_1, \ldots, w_k\}$, then $k \ge n$. o/w, $\{v_1, \ldots, v_n\}$ l dep.

Then $k = n$ #.

We have shown that if a vector space V has a basis with n elements, then any other basis must have n elements too.

Definition: In this case, we say that n is the **dimension** of V, and denote its dimension by $\dim V$.

Example 1: We have seen that $\mathbb{R}^n$ has a basis with n elements (the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$). $\mathbb{R}^n$ is n-dimensional, or $\dim \mathbb{R}^n = n$.

Example 2: Let $\mathbf{v}_1 = (1, 2, 3)^T$ and $\mathbf{v}_2 = (0, 1, 2)^T$, and $\mathbf{v}_3 = (0, 4, 8)^T$.

(1) What's dimension and basis of span$\{\mathbf{v}_1, \mathbf{v}_2\}$? $\{v_1, v_2\}$ l. indep

$\{v_1, v_2\}$ spans the whole space span$\{v_1, v_2\}$.

Then $\{v_1, v_2\}$ is a basis for span$\{v_1, v_2\}$.

$\dim(\text{span}\{v_1, v_2\}) = 2$.

(2) What's dimension and basis of span$\{\mathbf{v}_2, \mathbf{v}_3\}$?

$4v_2 = v_3$.

$\text{span}\{v_2, v_3\} = \text{span}\{v_2\}$.

A basis is $\{v_2\}$ (or $\{v_3\}$) $\dim(\text{span}\{v_2, v_3\}) = 1$

Example 3: Find a basis and the dimension of the following spaces:

(1) The vector space $\mathcal{P}^{(n)}$ of polynomials of degree $\leq n$.

① $P^{(n)} = \text{span}\{x^n, \ldots, x^1, 1\}$.

② $\{x^n, x^{n-1}, \ldots, x, 1\}$ are l. indep.

Thus, $\{x^n, \ldots, x^1, 1\}$ is a basis for $P^{(n)}$.

$\dim P^{(n)} = \underline{n+1}$

(2) The vector space $M_{2\times 2}(\mathbb{R})$, the set of all 2×2 matrices. $\underline{\dim(M_{2\times 2}) = 4}$ #

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A_3} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_4} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

Then ① $\{A_1, \ldots, A_4\}$ spans $M_{2\times 2}(\mathbb{R})$.

② $\{A_1, \ldots, A_4\}$ are l. indep. since $aA_1 + \cdots + dA_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\Rightarrow a = b = c = d = 0$. Thus, $\{A_1, \ldots, A_4\}$ is a basis.

(3) The vector space $M_{m\times n}(\mathbb{R})$.

Similar as (2).

$\dim(M_{m\times n}(\mathbb{R})) = \underline{mn}$ #

Example 4: Determining if $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for $\mathbb{R}^3$.

<u>EX 3 (4):</u> upper triangular matrix of 3×3 matrix

$$S = \left\{ \begin{bmatrix} a_1 & a_4 & a_5 \\ 0 & a_2 & a_6 \\ 0 & 0 & a_3 \end{bmatrix} \,\middle|\, a_1, \ldots, a_6 \in \mathbb{R} \right\}.$$

$$\underline{\dim S = 6}.$$