Lecture 12: Quick review from previous lecture

• If \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) are vectors in a vector space \( V \), we say they are **linearly dependent** if there exist scalars \( c_1, \ldots, c_n \), not all of which are zero, so that

\[
c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n = \mathbf{0}.
\]

If all \( c_i \) can only be zero, then we call \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) are **linearly independent**.

Today we will discuss

• Sec. 2.4 Basis and Dimension.

- Lecture will be recorded -
Fact 2: Let \( k \leq n \). A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) in \( \mathbb{R}^n \) is linearly independent if and only if the rank of \( A = [\mathbf{v}_1, \ldots, \mathbf{v}_k] \) is equal to \( k \).

\[
(\Leftarrow) \quad \text{Set up } c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{u}_k = 0. \quad \text{Show } c_1 = \cdots = c_k = 0.
\]

Since \( \text{rank}(A) = k \), we have

\[
A \xrightarrow{\text{row ops.}} U = \begin{bmatrix} a_1 & \cdots & a_k \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n \times k} = \begin{bmatrix} B \\ 0 \end{bmatrix}_{n \times k},
\]

where \( B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \) is indeed nonsingular.

\[
\Rightarrow \quad c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{u}_k = 0 \Rightarrow A \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow B \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\]

\[
(\Rightarrow) \quad \text{skip}.
\]

\[
\text{Since } B \text{ is nonsingular, } c_1 = \cdots = c_k = 0 \quad \text{since } B \text{ is nonsingular.}
\]

Fact 3: If \( \mathbf{v}_n \) can be written as a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \), then

\[
\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}.
\]

*See also Example 4: If \( \mathbf{v}_1 = c \mathbf{v}_2 \), then \( \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\} \).

It's clear to see \( \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \) (A \supseteq B: A contains B)

we only need to show \( \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \supseteq \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \).

For any \( \mathbf{w} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \), we can express \( \mathbf{w} \) as a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_n \), that is,

\[
\mathbf{w} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n.
\]

Since \( \mathbf{u}_n \) is a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \), we have

\[
\mathbf{u}_n = c_1 \mathbf{v}_1 + \cdots + c_{n-1} \mathbf{v}_{n-1} + c_n (a_1 \mathbf{v}_1 + \cdots + a_{n-1} \mathbf{v}_{n-1})
\]

\[
= (c_1 + a_1 c_n) \mathbf{v}_1 + \cdots + (c_{n-1} + a_{n-1} c_n) \mathbf{v}_{n-1}.
\]

Then \( \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \).

\( \checkmark \) \( \mathbf{v}_n \) is redundant in view of \( \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\} \).
2.4 Basis and Dimension

Definition:

(1) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we say that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $V$.

(2) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $V$ and are linearly independent, we say that they form a basis of a vector space $V$.

*So a basis for a vector space $V$ is a linearly independent set of vectors that span $V$.

Example 1.

(1) Check $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$, $\mathbf{e}_3 = (0,0,1)^T$ are linearly independent.

\[
\begin{bmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\Rightarrow a_1 = a_2 = a_3 = 0.
\]

(2) We have known that $\text{span}\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\} = \mathbb{R}^3$. Thus, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$.

In general, the “standard basis” of $\mathbb{R}^n$ consists of the $n$ vectors:

\[
\mathbf{e}_1 = 
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\mathbf{e}_2 = 
\begin{pmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \ldots, 
\mathbf{e}_n = 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
\]

Here $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent and they span $\mathbb{R}^n$, since any vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ can be written as $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i$. 

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A natural question is: can there be a basis of $\mathbb{R}^n$ with a different number of vectors (not $n$)?

The answer is “no”! In fact

**Fact 1:** Any basis of $\mathbb{R}^n$ must have exactly $n$ vectors.
In addition, a set of $v_1, \ldots, v_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if and only if $A = [v_1, \ldots, v_n]$ is nonsingular ($\text{rank}(A) = n$).

[To see this] From Fact 2 in Section 2.3, we have

$$\Rightarrow \text{ } \{v_1, \ldots, v_n\} \text{ is a basis } \Rightarrow \{v_1, \ldots, v_n\} \text{ l. ind.}$$

$$\Leftarrow \text{ } A = [v_1, \ldots, v_n] \text{ has rank } n$$

$$\Leftrightarrow \text{ } A \text{ is nonsingular.}$$

**Fact 2:** Let $V = \text{span}\{v_1, \ldots, v_n\}$ be a spanning set of $n$ linearly independent vectors, then any set of $k$ elements $w_1, \ldots, w_k$ in $V$ with $k > n$ is linearly dependent.

Now, we consider the fact

Then we can show the general case.

**Fact 3:** If $V$ is any vector space that has a basis with $n$ vectors, then any other basis must also have $n$ vectors.

[To see this]

Suppose $V$ has a basis $\{v_1, \ldots, v_n\}$, and also it has another basis $\{w_1, \ldots, w_k\}$.

Show $k = n$.

1. $V$ has a basis $\{v_1, \ldots, v_n\}$ ($V = \text{span}\{v_1, \ldots, v_n\}$), then $k \leq n$. $o/w \{w_1, \ldots, w_k\}$ are l. dep.

2. $V$ has a basis $\{w_1, \ldots, w_k\}$, then $k \geq n$. $o/w$, $\{v_1, \ldots, v_n\}$ l. dep.

Then $k = n$. #
We have shown that if a vector space $V$ has a basis with $n$ elements, then any other basis must have $n$ elements too.

**Definition:** In this case, we say that $n$ is the **dimension** of $V$, and denote its dimension by $\dim V$.

**Example 1:** We have seen that $\mathbb{R}^n$ has a basis with $n$ elements (the standard basis $e_1, \ldots, e_n$). $\mathbb{R}^n$ is $n$-dimensional, or $\dim \mathbb{R}^n = n$.

**Example 2:** Let $v_1 = (1, 2, 3)^T$ and $v_2 = (0, 1, 2)^T$, and $v_3 = (0, 4, 8)^T$.

1. What’s dimension and basis of $\text{span}\{v_1, v_2\}$?
   - $\text{span}\{v_1, v_2\}$ spans the whole space $\text{span}\{v_1, v_2\}$.
   - Then $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2\}$.
   - $\dim (\text{span}\{v_1, v_2\}) = 2$.
2. What’s dimension and basis of $\text{span}\{v_2, v_3\}$?
   - $4v_2 = v_3$.
   - $\text{span}\{v_2, v_3\} = \text{span}\{v_3\}$.
   - A basis is $\{v_2\}$ (or $\{v_3\}$) $\dim (\text{span}\{v_2, v_3\}) = 1$.

**Example 3:** Find a basis and the dimension of the following spaces:

1. The vector space $P^{(n)}$ of polynomials of degree $\leq n$.
   - $P^{(n)} = \text{span}\{x^n, \ldots, x, 1\}$.
   - $\{x^n, x^{n-1}, \ldots, x, 1\}$ are l. ind.
   - Thus, $\{x^n, \ldots, x, 1\}$ is a basis for $P^{(n)}$.
   - $\dim P^{(n)} = n + 1$.
(2) The vector space $M_{2 \times 2}(\mathbb{R})$, the set of all $2 \times 2$ matrices.

$$M_{2 \times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Then $\{A_1, \ldots, A_4\}$ spans $M_{2 \times 2}(\mathbb{R})$.

(3) The vector space $M_{m \times n}(\mathbb{R})$.

Similar as (2).

$$\dim \left( M_{m \times n}(\mathbb{R}) \right) = mn$$

Example 4: Determining if $v_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ form a basis for $\mathbb{R}^3$.

**EX 3 (4):** upper triangular matrix of $3 \times 3$, matrix

$$S = \left\{ \begin{bmatrix} a_1 & a_4 & a_5 \\ 0 & a_2 & a_6 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_1, \ldots, a_6 \in \mathbb{R} \right\}$$

$$\dim S = 6$$