Lecture 13: Quick review from previous lecture

- A **basis** $\{v_1, \dots, v_n\}$ for a vector space V is a 1 **linearly independent** set of vectors that 2 **span** V. $(V = \text{span}\{v_1, \dots, v_n\}$.
- $\mathbb{R}^n$ has a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. $\dim \mathbb{R}^n = n$.
- If V is any vector space that has a basis with n vectors, then any other basis must also have n vectors.

Today we will discuss

- Sec. 2.4 Dimension and basis
- Sec. 2.5 The fundamental matrix subspaces

- Lecture will be recorded -

- Exam 1 will cover C.1 and C.2, except 1.7, 2.5, 2.6.
- "Exam 1 instructions" have been posted on Canvas homepage (click Exam 1 Instructions). "Practice Exam" can be found there.
- Exam (2/24, Wed.) is **closed book** and everyone needs to **open camera**.

① span $\mathbb{R}^3$

② l. indep.

Example 4: Determining if $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for $\mathbb{R}^3$. We knew $\dim \mathbb{R}^3 = 3$. It's sufficient to check if $\{v_1, v_2, v_3\}$ is l. indep.

Set $a v_1 + b v_2 + c v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Find a, b, c.

$$\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ (homogeneous l. system)

If A is invertible, then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

To see if A is invertible,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 1 & 1 \\ 4 & 0 & 0 \end{bmatrix} \xrightarrow{①\leftrightarrow②} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 4 & 0 & 0 \end{bmatrix} \xrightarrow{③-2①} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{③-2②} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\det A = (-1)^{1} (2 \cdot (-1) \cdot (-2)) = \underline{-4} \neq 0.$$ A is nonsingular / invertible

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \#$$

Example 5: Check if $p_1(x) = 2x^2 + 4$, $p_2(x) = x^2 - x$, $p_3(x) = x^2$ form a basis for $\mathcal{P}^{(2)}$. $\dim P^{(2)} = 3$.

(Recall $\{x^2, x, 1\}$ is a basis for $P^{(2)}$). Set $a p_1 + b p_2 + c p_3 = 0$

$$a(2x^2 + 4) + b(x^2 - x) + c(x^2) = 0.$$

$$(\underbrace{2a + b + c}_{=0}) x^2 + (\underbrace{-b}_{=0}) x + \underbrace{4a}_{=0} = 0.$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 4 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\det A = 4 \neq 0$. Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

which implies $\{p_1, p_2, p_3\}$ is a basis for $P^{(3)}$.

Remark: So far we have seen that $\{1, x, x^2\}$ and $\{2x^2+4, x^2-x, x^2\}$ are basis for $\mathcal{P}^{(2)}$. View $q(x) = x^2+2x+6$ as a vector $\begin{pmatrix}1\\2\\6\end{pmatrix}$ in basis $\{x^2, x, 1\}$

a vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ in New basis $\{p_1, p_2, p_3\}$.

Question: How do we determine the coefficients of a quadratic polynomial in the basis $p_1(x) = 2x^2+4$, $p_2(x) = x^2-x$, $p_3(x) = x^2$?

For example,

Example 6: Consider $q(x) = x^2+2x+6$. Determine a, b, c in $q = ap_1 + bp_2 + cp_3$.

$$ap_1 + bp_2 + cp_3 = q$$

Similar as above

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 4 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

$$\left[A \,\middle|\, \begin{matrix}1\\2\\6\end{matrix}\right] \xrightarrow{③-2①} \left[\begin{matrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & -2 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 2 \\ 4 \end{matrix}\right] \xrightarrow{③-2②} \left[\begin{matrix} 2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 2 \\ 0 \end{matrix}\right].$$

③ $c = 0$

② $b = -2$

① $a = 3/2$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{\begin{pmatrix} 3/2 \\ -2 \\ 0 \end{pmatrix}}.$$

represents the vector of q in this New basis $\{p_1, p_2, p_3\}$

Actually this expression is unique.

Fact 4: The elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of a vector space V if and only if every $\mathbf{x} \in V$ can be written uniquely as a linear combination of the basis elements:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

#

$(\Longrightarrow)$ Since $\{v_1, \ldots, v_n\}$ is a basis, we have

$x = c_1 v_1 + \cdots + c_n v_n$

Suppose $x = a_1 v_1 + \cdots + a_n v_n$.

$0 = x - x = (c_1 - a_1) v_1 + \cdots + (c_n - a_n) v_n$

Since $\{v_1, \ldots, v_n\}$ are l. indep, $c_k - a_k = 0$, $1 \le k \le n$

$c_k = a_k \ , \ 1 \le k \le n.$ #

2.5 The Fundamental Matrix Subspaces

$A_{m\times n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$

§ Kernel and Image

We can associate to a matrix $A = A_{m\times n}$ a subspace of $\mathbb{R}^n$, called the *kernel* or *null space* of A.

$\ker A \quad \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

Definition: The **kernel** of A is the set of all solutions $\mathbf{x}$ to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. We denote the kernel of A by $\ker A$:

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Fact 1: $\ker A$ is a subspace of $\mathbb{R}^n$:

[To see this:]

(1) $A \underset{\mathbb{R}^n}{\underline{0}} = \underset{\mathbb{R}^m}{\underline{0}}$. So $\underset{\mathbb{R}^n}{0} \in \ker A$

(2) For $x, y \in \ker A$, check $\underline{x+y \in \ker A}$:

$$\begin{aligned} A(x+y) &= Ax + Ay \\ &= 0 + 0 \\ &= 0. \end{aligned} \quad \Big] \begin{aligned} Ax &= 0 \\ Ay &= 0 \end{aligned}$$

$\Rightarrow x+y \in \ker A.$

(3) $c \in \mathbb{R}$, $x \in \ker A$. check $\underline{cx \in \ker A}$:

$A(cx) = cAx = c0 = 0$. So $cx \in \ker A$.

Let's observe that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to the equation $A\mathbf{x} = \mathbf{b}$, then what can we say about their difference, $\mathbf{x}_1 - \mathbf{x}_2$?

$Ax_1 = b$

$Ax_2 = b$

Then $A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0.$

$$\boxed{x_1 - x_2 \in \ker A}$$

Fact 2: Suppose the linear system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}^\star$. Then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b} \iff \mathbf{x} = \mathbf{x}^\star + z$, where $z \in \ker A$.

[To see this:]

($\Rightarrow$) We know both x^*, x are solutions of $Ax=b$.
Based on observation above,

$$x - x^* \in \ker A.$$

Let $z = x - x^*$. Then $z \in \ker A$. So we get

$$x = \underbrace{\underline{x^*}}_{\text{sol. to } Ax=b} + \underbrace{\underline{z}}_{\in \ker A}$$

($\Leftarrow$)

$$\begin{aligned} A(x) &= A(x^* + z) \\ &= Ax^* + Az \\ &= b + 0 \\ &= b. \end{aligned}$$

$Ax^*=b$
$z \in \ker A$

This gives x is a solution to $Ax=b$.

Remark. In other words, *any* solution to $A\mathbf{x} = \mathbf{b}$ can be generated by starting with a *particular* solution $\mathbf{x}^*$, and adding to $\mathbf{x}^*$ vectors in the kernel of A.

$x^* + z$ $\ker A$

Example 1. Write the general solution to the following linear system in the form of $\mathbf{x} = \mathbf{x}^\star + z$, where $z \in \ker A$.

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix}}^{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ -4 \end{pmatrix}$$

$$\left(A \,\middle|\, \begin{matrix} -1 \\ -6 \\ -4 \end{matrix}\right) \xrightarrow[③-2①]{②-2①} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 1 & -2 & -2 \end{array}\right) \xrightarrow{③-\frac{1}{2}②} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Free variable : c

② $2b - 4c = -4 \Rightarrow b = 2c - 2$.

① $a - b = -1 \Rightarrow a = -1 + b$
$= 2c - 3$

all solutions are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c-3 \\ 2c-2 \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} c}_{\boxed{\ker A}} + \underbrace{\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}}_{x^*}.$$

x^*, a solution to $Ax=b$.

Fact 3: If A is an $m \times n$ matrix, then the following conditions are equivalent:

1. $\ker A = \{\mathbf{0}\}$ (homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$);
2. $\mathsf{rank}(A) = n$;
3. The linear system $A\mathbf{x} = \mathbf{b}$ has no free variables;
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \operatorname{img} A$.