Lecture 14: Quick review from previous lecture

① l. indep ② spans V

- The elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{x} \in V$ can be written uniquely as a linear combination of the basis elements:

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n.$$

- Let A be $m \times n$ matrix. The **kernel** of A is

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : \underbrace{A\mathbf{x} = \mathbf{0}}_{\text{homogeneous l. system}}\},$$

which is a subspace of $\mathbb{R}^n$.

Today we will discuss

- Sec. 2.5 the Kernel and Image.

- Lecture will be recorded -

- HW 5 due today 6pm.
- Exam 1 will cover C.1 and C. 2, except 1.7, 2.5, 2.6.
- "Exam 1 instructions" have been posted on Canvas homepage (click Exam 1 Instructions). "Practice Exam" can be found there.
- Exam (2/24, Wed.) is **closed book** and everyone needs to **open camera**.

Example 1. Write the general solution to the following linear system in the form of $\mathbf{x} = \mathbf{x}^\star + z$, where $z \in \ker A$.

$$\overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix}}^{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ -4 \end{pmatrix}$$

$$\left(A \,\middle|\, \begin{matrix} -1 \\ -6 \\ -4 \end{matrix}\right) \xrightarrow[③-2①]{②-2①} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 1 & -2 & -2 \end{array}\right) \xrightarrow{③-\frac{1}{2}②} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

[See Lecture 13]

<u>Free variable</u> : c

② $2b - 4c = -4 \Rightarrow \underline{b = 2c - 2}$.

① $a - b = -1 \Rightarrow a = -1 + b$
$= \underline{2c - 3}$

all solutions are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c - 3 \\ 2c - 2 \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} c}_{\boxed{\ker A}} + \underbrace{\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}}_{x^*}.$$

x^*, a solution to $Ax = b$.

Fact 3: If A is an $m \times n$ matrix, then the following conditions are equivalent:

1. $\ker A = \{\mathbf{0}\}$ (homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$);
2. $\mathsf{rank}(A) = n$;
3. The linear system $A\mathbf{x} = \mathbf{b}$ has no free variables;
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \operatorname{img} A$.

§ To construct a basis for $\ker A$.

Example 2. Suppose we have a 3-by-5 matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix}$.

Find a basis for $\ker A$. $= \{ x \in \mathbb{R}^5 \mid Ax = 0 \}$. $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix}$.

$$A \xrightarrow[③ - ①]{② - 2①} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

$\boxed{\text{rank } A = 3}$

Free variables : x_3, x_5

③ $x_4 = -2x_5$

② $x_2 = x_3 - 2x_5$

① $x_1 = -x_2 = -(x_3 - 2x_5) = -x_3 + 2x_5$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + 2x_5 \\ x_3 - 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix} x_5, \quad x_3, x_5 \in \mathbb{R}$$

$$\ker A = \left\{ \begin{pmatrix} -x_3 + 2x_5 \\ x_3 - 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} \;\middle|\; x_3, x_5 \in \mathbb{R} \right\}.$$

To find a basis

$x_3 = 1$, $x_5 = 0$ $\qquad$ $x_3 = 0$, $x_5 = 1$

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$\boxed{\dim \ker A = 2}$
$\quad \|$
$5 - 3$
$\quad \|$
$\quad 2$

Then $\{v_1, v_2\}$ is a basis for $\ker A$.

Remark. In general, here's how to build a basis for the kernel. There will be **one basis vector for each free variable**, which is constructed by setting that free variable to 1, and all the other free variables to 0.

In particular, this tells us that

If $A = A_{m\times n}$ and r is the rank of A, then

$$\mathsf{dim}(\ker A) = n - r$$

$$\mathbb{R}^n \xrightarrow{A_{m\times n}} \mathbb{R}^m$$

kerA imgA

Now we define

Definition: The **image** of the matrix $A = A_{m\times n}$ is the set containing of all images of A, that is,

$$\text{img}\, A = \{\underbrace{A\mathbf{x}}_{\text{outcome}} : \mathbf{x} \in \mathbb{R}^n\}.$$

Fact 4: img A is a subspace of $\mathbb{R}^m$ since it is the span of the columns of A.

[To see this] $A = [v_1 \ \cdots \ v_n]_{m\times n}$, v_i = ith columns of A.

For any $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$,

$$Ax = [v_1 \ \cdots \ v_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \cdots + x_n v_n.$$

(l. combination of $v_1, \ldots, v_n$)

$$\text{img} A = \{ Ax \mid x \in \mathbb{R}^n \}$$

$$= \{ x_1 v_1 + \cdots + x_n v_n \mid x_1, \ldots, x_n \in \mathbb{R} \}.$$

$= \text{span}\{ v_1, \ldots, v_n \}$. Then img A is a subspace of $\mathbb{R}^m$.

§ **To construct a basis for img A.**

Example 3. (1) Find the basis of img A, where
$$A = \begin{array}{ccccc} v_1 & v_2 & & v_3 & \\ \end{array}$$
$$A = \begin{pmatrix} 2 & 2 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 4 & 4 & 2 & 6 & 4 \end{pmatrix}$$

$$A \xrightarrow{④-2①} \begin{pmatrix} 2 & 2 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{rank } A = 3$$

Look at submatrix formed from columns with pivots

$$\tilde{U} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{U} = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} \text{ is nonsingular.}$$

So a basis of img A is $\{v_1, v_2, v_3\}$

(columns of "original" A corresponding to pivots)

(2) Find the basis of ker A as well. Free variables: x_3, x_5.

$$\begin{pmatrix} 2 & 2 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} :$$

③ : $4x_4 + 2x_5 = 0$

$x_4 = -\frac{1}{2}x_5$

② : $x_2 - x_4 + x_5 = 0$

$x_2 = x_4 - x_5 = -\frac{3}{2}x_5$

① : $2x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 = 0$

$x_1 = -\frac{1}{2}x_3 + \frac{5}{4}x_5$.

$$\ker A = \left\{ \begin{pmatrix} -\frac{1}{2}x_3 + \frac{5}{4}x_5 \\ -\frac{3}{2}x_5 \\ x_3 \\ -\frac{1}{2}x_5 \\ x_5 \end{pmatrix} \;\middle|\; x_3,\ x_5 \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} x_3 + \underbrace{\begin{pmatrix} \frac{5}{4} \\ -\frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}}_{v_2} x_5 \;\middle|\; x_3,\ x_5 \in \mathbb{R} \right\}$$

$\{v_1, v_2\}$ is a basis for $\ker A$. #

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 0 \\ -2 \\ 4 \end{pmatrix} \right\}.$$

This same reasoning will work in general.

Fact 5:

- To find a basis for img A, bring A to "row echelon form" by Gaussian elimination. Then **the columns** of A (the original matrix) where the **pivots** occur form **a basis for img** A.
- In particular, the **dimension of the image of** A will be the number of pivots, i.e. the rank r of A. That is,

$$\mathsf{dim}(\operatorname{img} A) = r.$$

* Note that the rank of r does not depend on how we perform Gaussian elimination (that is, which permutations we perform). rank(A) only depends on A itself.

So far, we have seen that

If $\mathsf{rank} A_{m\times n} = r$, then

$$\mathsf{dim}(\operatorname{img} A) = r, \quad \mathsf{dim}(\ker A) = n - r$$

$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$ $\qquad$ $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

$\mathbb{R}^n \xleftarrow{A^T} \mathbb{R}^m$ $\qquad$ $\ker A$ $\qquad$ $\operatorname{img} A$

§ Coimage of A and cokernel of A.

$\boxed{\operatorname{img} A^T}$ $=$ coimg A $\boxed{\ker A^T}$ $=$ coker A

Definition: The **coimage** of A is the image of its transpose, A^T. It is denoted coimg A:

$$\operatorname{coimg} A = \operatorname{img} A^T = \{A^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

The **cokernel** of A is the kernel of its transpose, A^T. It is denoted coker A:

$$\operatorname{coker} A = \ker A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T\mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

✓ img A, ker A, coimg A, and coker A are the four fundamental "subspaces" of A.

§ coimg A

Let's study coimg A first. It is the **span of the columns of A^T**, or equivalently the span of the **rows** of A. For this reason, it's also called the $\underbrace{\textbf{row space of } A}_{\operatorname{coimg} A}$.

$$A = \begin{bmatrix} w_1^T \\ \vdots \\ w_m^T \end{bmatrix}_{m\times n} \qquad A^T = \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}_{n\times m}$$

$$\operatorname{coimg} A = \operatorname{img}(A^T) = \operatorname{span}\{w_1 \cdots w_m\} = \operatorname{span}\{\text{rows of } A\}$$

In principle, we could build a basis for coimg A the same way we learned how to do for img A, by performing Gaussian elimination on A^T and taking the columns of A^T with pivots.

Similar to what we did in **Example 3** for A.

Example 4: We first bring A^T down to the row echelon form of A^T. Suppose we know the row echelon form of A^T:

$$A^T = \begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 7 & 2 \\ 2 & 4 & 4 & 0 & -2 \end{pmatrix} \underbrace{\longrightarrow}_{\textit{row operations}} \begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then a basis of coimg (A) (img (A^T)) is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}.$

However, there is an **alternate way** of building **a basis for coimg** A. This method will let us see a profound connection between "img A" and "coimg A".

Observation.

- Performing for Gaussian elimination: (a) adding a multiple of one row to another row; and (b) permuting the order of rows, we have

$$A \underbrace{\longrightarrow}_{row\,operations} U \text{ (row echelon form)}$$

- Both row operations (a) and (b) above obviously do **not** change the row span (the row space of A).

Consequently, we have

Conclusion 1: The row echelon matrix U has exactly the same row space as the original matrix A.

Conclusion 2: Therefore, we can construct **a basis for the (row space)**

coimg A

by bringing A to row echelon form using Gaussian elimination, and take the nonzero rows as the basis vectors.

Poll Question 1: Let V be a vector space with $\dim V = 5$ (dimension of V). Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_4$ in V are linearly independent, can $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ form a basis for V?

A) Yes

B) No