

Lecture 14: Quick review from previous lecture

① l. indep ② spans V

- The elements $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of V if and only if every $\mathbf{x} \in V$ can be written **uniquely** as a linear combination of the basis elements:

$$\mathbf{x} = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n.$$

- Let A be $m \times n$ matrix. The **kernel** of A is

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{0}\},$$

which is a **subspace** of \mathbb{R}^n .

homogeneous l. system

Today we will discuss

- Sec. 2.5 the Kernel and Image.

- Lecture will be recorded -

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- HW 5 due today 6pm.
 - Exam 1 will cover C.1 and C. 2, except 1.7, 2.5, 2.6.
 - “Exam 1 instructions” have been posted on Canvas homepage (click [Exam 1 Instructions](#)). “Practice Exam” can be found there.
 - Exam (2/24, Wed.) is **closed book** and everyone needs to **open camera**.

Example 1. Write the general solution to the following linear system in the form of $\mathbf{x} = \mathbf{x}^* + z$, where $z \in \ker A$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -4 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ -4 \end{pmatrix}$$

$$\left(A \mid \begin{matrix} -1 \\ -6 \\ -4 \end{matrix} \right) \xrightarrow[\textcircled{3} -2 \textcircled{1}]{\textcircled{2} -2 \textcircled{1}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 1 & -2 & -2 \end{array} \right) \xrightarrow{\textcircled{3} -\frac{1}{2} \textcircled{2}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

[See Lecture 13]

Free variable = c

$$\textcircled{2} \quad 2b - 4c = -4 \Rightarrow \underline{b = 2c - 2}$$

$$\textcircled{1} \quad a - b = -1 \Rightarrow a = -1 + b = \underline{2c - 3}$$

all solutions are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c - 3 \\ 2c - 2 \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} c + \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$$

ker A

\mathbf{x}^* , a solution to $A\mathbf{x} = \mathbf{b}$.

Fact 3: If A is an $m \times n$ matrix, then the following conditions are equivalent:

1. $\ker A = \{\mathbf{0}\}$ (homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$);
2. $\text{rank}(A) = n$;
3. The linear system $A\mathbf{x} = \mathbf{b}$ has no free variables;
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \text{img } A$.

§ To construct a basis for $\ker A$.

Example 2. Suppose we have a 3-by-5 matrix $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix}$.

Find a basis for $\ker A = \{x \in \mathbb{R}^5 \mid Ax = 0\}$. $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

$$A \xrightarrow[\text{(3) - (1)}]{\text{(2) - 2(1)}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad \boxed{\text{rank } A = 3}$$

Free variables = x_3 , x_5

③ $x_4 = -2x_5$

② $x_2 = x_3 - 2x_5$

① $x_1 = -x_2 = -(x_3 - 2x_5) = -x_3 + 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 + 2x_5 \\ x_3 - 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix} x_5, \quad x_3, x_5 \in \mathbb{R}$$

$$\ker A = \left\{ \begin{pmatrix} -x_3 + 2x_5 \\ x_3 - 2x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} \mid x_3, x_5 \in \mathbb{R} \right\}$$

To find a basis

$x_3 = 1, x_5 = 0$

$x_3 = 0, x_5 = 1$

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \dim \ker A = 2 \\ \parallel \\ 5 - 3 \\ \parallel \\ 2 \end{array}}$$

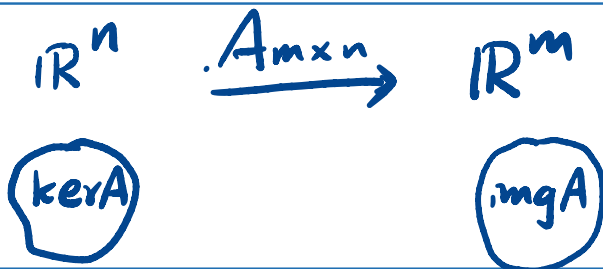
Then $\{v_1, v_2\}$ is a basis for $\ker A$.

Remark. In general, here's how to build a basis for the kernel. There will be **one basis vector for each free variable**, which is constructed by setting that free variable to 1, and all the other free variables to 0.

In particular, this tells us that

If $A = A_{m \times n}$ and r is the rank of A , then

$$\dim(\ker A) = n - r$$



Now we define

Definition: The **image** of the matrix $A = A_{m \times n}$ is the set containing of all images of A , that is,

$$\text{img } A = \{ \underline{Ax} : x \in \mathbb{R}^n \}.$$

outcome

Fact 4: $\text{img } A$ is a subspace of \mathbb{R}^m since it is the span of the columns of A .

[To see this] $A = [v_1 \ \dots \ v_n]_{m \times n}$, $v_i = i$ th columns of A .

For any $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in \mathbb{R}^n ,

$$Ax = [v_1 \ \dots \ v_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \dots + x_n v_n. \quad (\text{l. combination of } v_1, \dots, v_n)$$

$$\begin{aligned} \text{img } A &= \{ Ax \mid x \in \mathbb{R}^n \} \\ &= \{ x_1 v_1 + \dots + x_n v_n \mid x_1, \dots, x_n \in \mathbb{R} \}. \end{aligned}$$

MATH 4242-Week 5-5 $= \text{span} \{ v_1, \dots, v_n \}$. Then $\text{img } A$ is Spring, 2021

a subspace of \mathbb{R}^m .

§ To construct a basis for $\text{img } A$.

Example 3. (1) Find the basis of $\text{img } A$, where $A =$

$$A = \begin{pmatrix} \overset{v_1}{2} & \overset{v_2}{2} & 1 & \overset{v_3}{3} & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ \underset{4}{4} & 4 & 2 & 6 & 4 \end{pmatrix}$$

$$A \xrightarrow{\textcircled{4} - 2\textcircled{1}} \begin{pmatrix} \textcircled{2} & 2 & 1 & 3 & 2 \\ 0 & \textcircled{1} & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

, $\text{rank } A = 3$

Look at submatrix formed from columns with pivots

$$\tilde{U} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \hat{U} = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{pmatrix} \text{ is nonsingular.}$$

So a basis of $\text{img } A$ is $\{v_1, v_2, v_3\}$
 (columns of "original" A corresponding to pivots)

(2) Find the basis of $\text{ker } A$ as well.

Free variables: x_3, x_5 .

$$\begin{pmatrix} \textcircled{2} & 2 & 1 & 3 & 2 \\ 0 & \textcircled{1} & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} :$$

③ : $4x_4 + 2x_5 = 0$

$x_4 = -\frac{1}{2}x_5$

② : $x_2 - x_4 + x_5 = 0$

$x_2 = x_4 - x_5 = -\frac{3}{2}x_5$

① : $2x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 = 0$

$x_1 = -\frac{1}{2}x_3 + \frac{5}{4}x_5$

$$\ker A = \left\{ \begin{pmatrix} -\frac{1}{2}x_3 + \frac{5}{4}x_5 \\ -\frac{3}{2}x_5 \\ x_3 \\ -\frac{1}{2}x_5 \\ x_5 \end{pmatrix} \mid x_3, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} \frac{5}{4} \\ -\frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} x_5 \mid x_3, x_5 \in \mathbb{R} \right\}$$

$\{v_1, v_2\}$ is a basis for $\ker A$. #

This same reasoning will work in general.

Fact 5:

- To find a basis for $\text{img } A$, bring A to “row echelon form” by Gaussian elimination. Then **the columns** of A (the original matrix) where the **pivots** occur form a **basis for $\text{img } A$** .
- In particular, the **dimension of the image of A** will be the number of pivots, i.e. the rank r of A . That is,

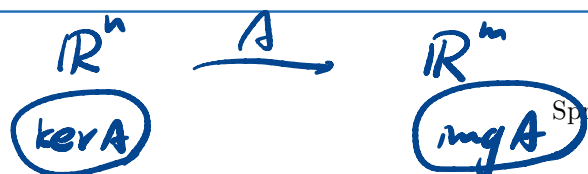
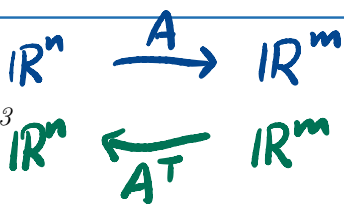
$$\dim(\text{img } A) = r.$$

* Note that the rank of r does not depend on how we perform Gaussian elimination (that is, which permutations we perform). $\text{rank}(A)$ only depends on A itself.

So far, we have seen that

If $\text{rank } A_{m \times n} = r$, then

$$\dim(\text{img } A) = r, \quad \dim(\ker A) = n - r$$



§ Coimage of A and cokernel of A .

$\text{img } A^T$
coimg A

$\text{ker } A^T$
coker A

Definition: The **coimage** of A is the image of its transpose, A^T . It is denoted $\text{coimg } A$:

$$\text{coimg } A = \text{img } A^T = \{A^T \mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

The **cokernel** of A is the kernel of its transpose, A^T . It is denoted $\text{coker } A$:

$$\text{coker } A = \text{ker } A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T \mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

✓ $\text{img } A$, $\text{ker } A$, $\text{coimg } A$, and $\text{coker } A$ are the four fundamental “subspaces” of A .

§ $\text{coimg } A$

Let’s study $\text{coimg } A$ first. It is the **span of the columns of A^T** , or equivalently the span of the **rows of A** . For this reason, it’s also called the **row space of A** .

$$A = \begin{bmatrix} w_1^T \\ \vdots \\ w_m^T \end{bmatrix}_{n \times m} \quad A^T = [w_1 \ \dots \ w_m]_{m \times n}$$

coimg A

$$\text{coimg } A = \text{img } (A^T) = \text{span} [w_1 \ \dots \ w_m] = \text{span} [\text{rows of } A]$$

In principle, we could build a basis for $\text{coimg } A$ the same way we learned how to do for $\text{img } A$, by performing Gaussian elimination on A^T and taking the columns of A^T with pivots.

Similar to what we did in **Example 3** for A .

Example 4: We first bring A^T down to the row echelon form of A^T . Suppose we know the row echelon form of A^T :

$$A^T = \begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 7 & 2 \\ 2 & 4 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then a basis of $\text{coimg}(A)$ ($\text{img}(A^T)$) is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}$$

However, there is an **alternate way** of building a **basis for $\text{coimg } A$** . This method will let us see a **profound connection** between “ $\text{img } A$ ” and “ $\text{coimg } A$ ”.

Observation.

- Performing for Gaussian elimination: (a) adding a multiple of one row to another row; and (b) permuting the order of rows, we have

$$A \xrightarrow[\text{row operations}]{} U \text{ (row echelon form)}$$

- Both row operations (a) and (b) above obviously do **not** change the row span (the **row space of A**).

Consequently, we have

Conclusion 1: The row echelon matrix U has exactly the **same row space** as the original matrix A .

Conclusion 2: Therefore, we can construct a **basis for the (row space) $\text{coimg } A$**

by bringing A to row echelon form using Gaussian elimination, and take the **nonzero rows as the basis vectors**.

Poll Question 1: Let V be a vector space with $\dim V = 5$ (dimension of V). Suppose $\mathbf{v}_1, \dots, \mathbf{v}_4$ in V are linearly independent, can $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ form a basis for V ?

A) Yes

B) No