Lecture 15: Quick review from previous lecture

Let A be an $m \times n$ matrix.

- The **kernel** of A is

$$\ker A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

- The **image** of the matrix A is the set containing of all images of A, that is,

$$\operatorname{img} A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \operatorname{span}\{\text{columns of } A\}$$

- The **coimage** of A is the image of its transpose, A^T. It is denoted coimg A:

$$\operatorname{coimg} A = \operatorname{img} A^T = \{A^T\mathbf{y} : \mathbf{y} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$
$$= \operatorname{span}\{\text{columns of } A^T\} = \operatorname{span}\{\text{rows of } A\}$$

- The **cokernel** of A is the kernel of its transpose, A^T. It is denoted coker A:

$$\operatorname{coker} A = \ker A^T = \{\mathbf{w} \in \mathbb{R}^m : A^T\mathbf{w} = \mathbf{0}\} \subset \mathbb{R}^m$$

Today we will

- continue discussing Sec. 2.5 the kernel and image, coker, and coimg.
- discuss Sect. 3.1 Inner Products

- Lecture will be recorded -

- Exam (2/~~14~~24, Wed.) is **closed book** and everyone needs to **open camera**.
- During the exam, you can see Exam 1 problems through
  1) Canvas:

  Assignments → Exam 1

  2) instructor's share screen via Zoom (contains first couple of problems due to the limit of screen).

However, there is an **alternate way** of building **a basis for coimg** A. This method will let us see a profound connection between "img A" and "coimg A".

Observation.

- Performing for Gaussian elimination: (a) adding a multiple of one row to another row; and (b) permuting the order of rows, we have

$$\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}_{m\times n} = A \underbrace{\longrightarrow}_{row\ operations} U \text{ (row echelon form)} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

- Both row operations (a) and (b) above obviously do **not** change the row span (the row space of A)

coimg A = span$\{w_1, \dots, w_m\}$ = span$\{u_1, \dots, u_m\}$.

Consequently, we have

Conclusion 1: The row echelon matrix U has exactly the same row space as the original matrix A.

Conclusion 2: Therefore, we can construct **a basis for the (row space)**

coimg A

by bringing A to row echelon form using Gaussian elimination, and take the nonzero rows as the basis vectors.

Example 5. The same matrix as in Example 3:

$$A \longrightarrow U = \begin{pmatrix} 1 & 2 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then a basis of coimg (A) (img (A^T)) is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 2 \end{pmatrix} \right\}$.

Fact 6: If the rank of A is r, the basis we construct for coimg A will have r vectors. Thus,

$$\mathsf{dim}(\text{img}\,A) = \mathsf{dim}(\text{coimg}\,A) = r$$

img A = span{columns of A}

coimg A = span{rows of A}

§ coker A

To build a basis for coker A, solve the n-by-m homogeneous system $A^T\mathbf{y} = \mathbf{0}$, and set each free variable to 1, and the others to zero.

*In other words, apply the exact same procedure as for finding a basis for ker A.

Q: What is the dimension of coker A?

It is the number of free variables in $A^T\mathbf{y} = \mathbf{0}$. Since A^T has m columns and rank r, there are $m - r$ free variables, hence

Fact 7: If A is an $m \times n$ matrix with $\mathsf{rank}(A) = r$, then

$$\mathsf{dim}(\text{coker}\,A) = m - r$$

Summary:

We can summarize what we've learned about the four fundamental subspaces in the following theorem, called the *Fundamental Theorem of Linear Algebra*:

Let A be an $m \times n$ matrix, and let r be its rank. Then

$$\dim \text{coimg}\,A = \dim \text{img}\,A = \text{rank}\,A = \text{rank}\,A^T = r,$$

$$\dim \ker A = n - r, \qquad \dim \text{coker}\,A = m - r.$$

**Again, a very useful (and surprising) aspect of this theorem is that the column space and row space of A have the same dimension, equal to the rank r of A.

Summary

Let A be an $m \times n$ matrix with $rank(A) = r$.

$$A \underbrace{\longrightarrow}_{row\, operations} U \text{ (row echelon form)}$$

dim	Vector Space	Basis
n-r	ker(A)	Solve $A\mathbf{x} = \mathbf{0}$, each free variable gives a basis vector
r	img (A)	columns of A where the pivots occur
r	coimg (A)	(1) columns of A^T where the pivots occur or (2) nonzero rows of U containing pivots
m-r	coker (A)	Solve $A^T\mathbf{x} = \mathbf{0}$, each free variable gives a basis vector

Example 4: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 4 \\ 1 & 0 & 1 & 1 \end{pmatrix}_{3\times 4}$$

Find a basis for ker A, img A, coimg A, coker A, respectively.

$$A \xrightarrow[③-①]{②-2①} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & -1 & -2 \end{pmatrix} \xrightarrow{③-②} \overset{U}{\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \text{ rank } A = 2$$

① A basis for img A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. dim(img A) = 2

② " for coimg A: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ -2 \end{pmatrix} \right\}$, dim(coimg A) = 2

③ A basis for ker A: dim(ker A) = $4 - 2 = 2$

Find x so that $Ax = 0$. Free variables x_3, x_4

(2): $-x_2 - x_3 - 2x_4 = 0$.

$x_2 = -x_3 - 2x_4$.

(1): $x_1 + x_2 + 2x_3 + 3x_4 = 0$

$$x_1 = -x_3 - x_4$$

$$\ker A = \left\{ \begin{pmatrix} -x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_3, x_4 \in \mathbb{R} \right\}.$$

$x_3 = 1, \; x_4 = 0$ $\qquad$ $x_3 = 0, \; x_4 = 1$

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$ is a basis for $\ker A$.

(4) A basis for coker $A = \ker(A^T)$:

dim coker A $= m - r = 1$.

$A^T x = 0$

exercise

$A^T \longrightarrow \begin{pmatrix} \quad \end{pmatrix}$, $\qquad$ coker $A = \left\{ \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} \middle| x_3 \in \mathbb{R} \right\}$.

A basis for coker A is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. #.

$$\mathbb{R}^n \xrightarrow{A_{m \times n}} \mathbb{R}^m$$

$\ker A$ $\qquad\qquad$ img A

coimg A $\qquad\qquad$ coker A

3 Inner Products and Norms

3.1 Inner Products

§ Inner products in the Euclidean space $\mathbb{R}^n$

Definition: If $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ are any two vectors in $\mathbb{R}^n$, then we define their **inner product**, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$, by:

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

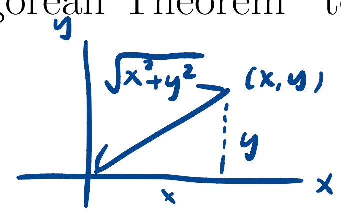
$(y_1, \cdots y_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Note that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x} \quad (\; = \mathbf{x}^T \mathbf{y})$$

$(x_1, \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

As in $\mathbb{R}^2$, if $\mathbf{x} = (x, y)^T$ is a vector, then the "Pythagorean Theorem" tells us that its length is given by $\sqrt{x^2 + y^2}$, and is denoted by



$$\|\mathbf{x}\| = \sqrt{x^2 + y^2}.$$

$= \sqrt{\langle \vec{x}, \vec{x} \rangle}$

Definition: We will use this to define the length of vectors in $\mathbb{R}^n$ and denote the length of a vector $\mathbf{x}$ by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + \ldots + x_n^2}$$

We call $\|\mathbf{x}\|$ the **norm** of $\mathbf{x}$.

If $\mathbf{x} \neq 0$, then $\|\mathbf{x}\| > 0$. In addition, we also have

$$\|\mathbf{x}\| = 0 \quad \Leftrightarrow \quad \mathbf{x} = 0.$$

Example 1. If $\mathbf{x} = (1,0,1)^T$ and $\mathbf{y} = (-2,1,2)^T$, then
(1) find $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, $\langle \mathbf{x}, \mathbf{x}+3\mathbf{y}\rangle$ and also normalize $\mathbf{x}$ and $\mathbf{y}$.
(2) Is $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$?

(1) $\|x\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$.

$\|y\| = \sqrt{(-2)^2+1^2+2^2} = \sqrt{9} = 3$.

normalize x, y.

$$\frac{x}{\|x\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \frac{y}{\|y\|} = \frac{1}{3}\begin{pmatrix}-2\\1\\2\end{pmatrix}.$$

$$\langle x, x+3y\rangle = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} + 3\begin{pmatrix}-2\\1\\2\end{pmatrix}\right\rangle$$

$$= \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-5\\3\\7\end{pmatrix}\right\rangle = -5+0+7$$

$$= \underline{2}$$

$$\|$$

$$\langle x, x\rangle + 3\langle x, y\rangle = \|x\|^2 + 3\left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\2\end{pmatrix}\right\rangle$$

$$= 2 + 3\cdot 0 = \underline{2}.$$

(2) Yes.

Q: Based on these observation on $\langle \mathbf{x}, \mathbf{y}\rangle$ on $\mathbb{R}^n$ above, what properties do you think they should hold for a "general" inner product.

§ Abstract definition of general inner products

Definition: Let V be a vector space. An **inner product** on V is a functions that assigns every pairing two vectors $\mathbf{x}$ and $\mathbf{y}$ in V to obtain a real number, denoted

$$\langle \mathbf{x}, \mathbf{y} \rangle,$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in \mathbb{R}$, the following hold:

(1) **Bilinearity:**

$$\langle c\mathbf{u} + d\mathbf{v},\ \mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{w} \rangle + d\langle \mathbf{v},\ \mathbf{w} \rangle,$$
$$\langle \mathbf{u},\ c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{v} \rangle + d\langle \mathbf{u},\ \mathbf{w} \rangle,$$

(2) **Symmetry:** $\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{w},\ \mathbf{v} \rangle$,

(3) **Positivity:** $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$. Moreover, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

Definition: A vector space V equipped with a specific inner product is called an **inner product space**.

The associate **norm** of a vector $\mathbf{v} \in V$ is defined as

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}. \quad (\text{By } (3),\ \|\mathbf{v}\| \geq 0).$$

In other words, an inner product space V that is a vector space equipped with an additional way of pairing two vectors $\mathbf{x}$ and $\mathbf{y}$ in V to obtain a real number, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$.

vector space + "inner product" $\Rightarrow$ inner product space

Example 2. We have known that the inner product on $\mathbb{R}^n$ defined earlier by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

satisfies these three axioms.

Example 3. Show that for all vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V,

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle \overset{(1)}{=} \langle x, x+y\rangle + \underline{\langle y, x+y\rangle}$$

$$\overset{(1)}{=} \underbrace{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$\overset{(2)}{=} \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

$\|x-y\|^2 =$

To be continued!

§ **The same vector space V can have many different inner products.** For example, while we originally equipped $\mathbb{R}^n$ with the standard inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n x_i y_i$, we can also define "other" inner products on $\mathbb{R}^n$ as well. See discussions below.

Example 4.(Another inner products on $\mathbb{R}^n$) If $c_1, \ldots, c_n$ are positive numbers, we can define

$$\langle \mathbf{x}, \mathbf{y}\rangle = c_1 x_1 y_1 + \ldots + c_n x_n y_n = \sum_{i=1}^n c_i x_i y_i. \tag{1}$$

This is a legitimate inner product (check this as an exercise). It is called a **weighted inner product**, with weights $c_1, \ldots, c_n$.

Observe that while we can write the ordinary inner product on $\mathbb{R}^n$ as $\mathbf{x}^T\mathbf{y}$, we can write the above **weighted inner product** as