Lecture 16: Quick review from previous lecture

- A a vector space V equipped with a specific inner product is called an **inner product space**.
- $\langle \mathbf{v},\ \mathbf{w} \rangle$ is an **inner product** on V if the following hold:

  (1) Bilinearity:

$$\langle c\mathbf{u} + d\mathbf{v},\ \mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{w} \rangle + d\langle \mathbf{v},\ \mathbf{w} \rangle,$$
$$\langle \mathbf{u},\ c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{v} \rangle + d\langle \mathbf{u},\ \mathbf{w} \rangle,$$

  (2) Symmetry: $\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{w},\ \mathbf{v} \rangle$,

  (3) Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$. Moreover, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

- Every inner product gives a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, that is, $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$

Today we will discuss

- Sec. 3.1 Inner Products and Norms.
- Sec. 3.2 Inequalities.

- Lecture will be recorded -

- HW 5 due today 6pm.
- The solutions, statistic, and grade for Exam 1 were posted on Canvas.

Example 3. Show that for all vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V,

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

$$\|x+y\|^2 = \langle \underline{x}+\underline{y}, x+y \rangle \overset{(1)}{=} \langle x, x+y\rangle + \langle y, x+y\rangle$$

$$\overset{(1)}{=} \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle.$$

$$\overset{(2)}{=} \langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle.$$

$$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

Similarly

$$\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2.$$

Then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \ \#.$$

§ **The same vector space V can have many different inner products.** For example, while we originally equipped $\mathbb{R}^n$ with the standard inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n x_i y_i$, we can also define "other" inner products on $\mathbb{R}^n$ as well. See discussions below. → Dot product.

Example 4.(Another inner products on $\mathbb{R}^n$) If $c_1, \ldots, c_n$ are positive numbers, we can define

$$\langle \mathbf{x}, \mathbf{y}\rangle = c_1 x_1 y_1 + \ldots + c_n x_n y_n = \sum_{i=1}^n c_i x_i y_i. \tag{1}$$

This is a legitimate inner product (check this as an exercise). It is called a **weighted inner product**, with weights $\underline{c_1}, \ldots, \underline{c_n}$.

① Bilinearity
② $\langle x, y\rangle = \langle y, x\rangle$
③ positivity.

Observe that while we can write the ordinary inner product on $\mathbb{R}^n$ as $\mathbf{x}^T\mathbf{y}$, we can write the above **weighted inner product** as

(usual inner product). $\langle x, y\rangle = x^T y = x^T \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} y$

(weight inner product)

$$\langle x, y \rangle = c_1 x_1 y_1 + \cdots + c_n x_n y_n$$
$$= (x_1, \cdots, x_n) \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$
$$= x^T D y, \text{ where } D = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix} = \operatorname{diag}(c_1, \cdots, c_n)$$

Q: Why do we require those $c_1 > 0, \ldots, c_n > 0$ in the inner product (1)?

$c_1 > 0, \ldots, c_n > 0$ is to ensure (3) positivity.

§ We can define an even more general class of inner products on $\mathbb{R}^n$, as follows:

Example 5. Take any n-by-n, nonsingular matrix A.

Now we define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \underbrace{A^T A}_{} \mathbf{y}. \quad — \quad (\Phi)$$

Let's check that this is an inner product. For $x, y, z \in \mathbb{R}^n$, $c, d \in \mathbb{R}$,

(1) Bilinearity:

$$\langle cx + dy, z \rangle \overset{(\Phi)}{=} (cx + dy)^T A^T A z$$
$$= (c x^T + d y^T) A^T A z.$$
$$= c(x^T A^T A z) + d(y^T A^T A z.)$$
$$= c \langle x, z \rangle + d \langle y, z \rangle.$$

Similarly

$$\langle x, cy + dz \rangle = \cdots \quad \cdots.$$

(2) Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.

$$\langle x, y \rangle = x^T A^T A y$$

Just a number.

$(x^T A^T A y)^T = (x^T A^T A y)$.

Ex: $3^T = 3$.

$$= (x^T A^T A y)^T$$
$$= y^T A^T A x$$

$$= \langle y, x \rangle.$$

(3) positivity:

$$\langle x, x\rangle = x^T A^T A x = (Ax)^T A x = z^T z = z_1^2 + \cdots + z_n^2 \geq 0.$$

$z = Ax = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$.

$\langle x, x\rangle = 0 \iff z_1^2 + \cdots + z_n^2 = 0 \iff z = 0 \iff Ax = 0 \iff x = 0$ since A is nonsingular.

Example 5. As an example of this kind of inner product in $\mathbb{R}^2$, let's define the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle = \underbrace{(x_1\ x_2)}_{x^T} \underbrace{\begin{pmatrix} \sqrt{2} & -\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix}}_{A^T} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{3} & \sqrt{3} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y} \tag{2}$$

$$= 2x_1y_1 + 3(x_2 - x_1)(y_2 - y_1) \tag{3}$$

$(1, 3) \underbrace{\begin{pmatrix} \sqrt{2} & -\sqrt{3} \\ 0 & \sqrt{3} \end{pmatrix}}_{A^T} = (\sqrt{2}\ \ 2\sqrt{3})$

Remark. [Comparison between the usual inner product and the one defined in (2):]

1. The norm of $(1, 3)^T$:

   - in usual inner product: $\|(1,3)^T\| = \sqrt{1^2 + 3^2} = \sqrt{10}$.

   $\|x\| = \sqrt{\langle x, x\rangle}$

   - in inner product defined in (2):

   $$\|(1,3)^T\| = \sqrt{\left\langle \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\vec{x}}, \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\vec{x}} \right\rangle} \overset{(2)}{=} \sqrt{(1, 3) A^T A \begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \sqrt{(\sqrt{2}\ \ 2\sqrt{3}) \begin{pmatrix} \sqrt{2} \\ 2\sqrt{3} \end{pmatrix}} = \sqrt{14}.$$ #

2. The inner product of $(1, 1)^T$ and $(-1, 2)^T$:

   - in usual inner product: $\langle (1,1)^T, (-1,2)^T \rangle = -1 + 2 = 1$

   - in inner product defined in (2):

   $$\langle (1, 1)^T, (-1, 2)^T \rangle \overset{(2)}{=} (1, 1) A^T A \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -2$$ #

$C^1(I) = \{ f \mid f, f' \text{ is continuous} \}$

Example 6. Let $C^0 = C^0(I)$ denote the vector space of continuous functions on an interval $I = [a, b]$, with the usual addition and scalar multiplication operations.

We can turn C^0 into an "**inner product space**" by defining the following **inner product**:

For any $f, g \in C^0(I)$,

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

*This is sometimes called the L^2 inner product (the "L" stands for "Lebesgue").

Let's check that this satisfies the defining properties of an inner product:

(1) Bilinearity : exercise.

(2) symmetry :

(3) positivity : $\langle f, f \rangle = \int_a^b f(x) f(x)\, dx = \int_a^b f^2 dx \geq 0.$

show : $\langle f, f \rangle = 0 \iff f = 0$

($\Leftarrow$) clearly.

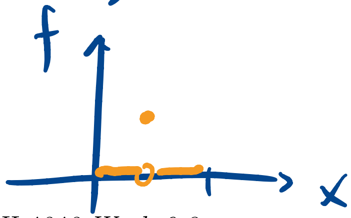
($\Rightarrow$) $\langle f, f \rangle = 0 \Rightarrow \int_a^b f^2 dx = 0.$

since $f^2 \geq 0$ and f^2 is continuous,

the function f must be zero everywhere.

o/w $\int_a^b f^2 \neq 0.$

Q: why "continuous" is important here?



f is NOT continuous, but $\int_0^1 f^2 dx = 0.$

$f \not\equiv 0.$

Summary.

- We saw many different inner products on $\mathbb{R}^n$, namely those of the form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T B \mathbf{y}$$

  for a matrix $B = A^T A$ where A is nonsingular.

- When $B = I$, this includes the usual inner product $\sum_{i=1}^n x_i y_i$. Note that this usual inner product on $\mathbb{R}^n$ is called the *dot product*.

- Let's look at another example with weighted inner product.

Example 7. We can also define **weighted inner products** on $C^0(I)$. If $w(x)$ is any positive, continuous function, we can define

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx \tag{4}$$

It is a straightforward exercise to check that this is an inner product.

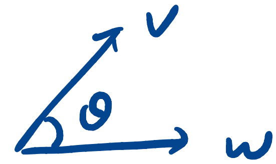
Thus $C^0(I)$ is also an **inner product space**, equipped with inner product (4).

3.2 Inequalities

§ The Cauchy-Schwarz Inequality

Let's first recall the result in $\mathbb{R}^n$.

In $\mathbb{R}^n$, we have seen from Calculus that the dot product between two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ can be geometrically characterized by



$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

where θ is the angle between the two vectors. Thus, $|\cos\theta| \leq 1.$

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$ (Cauchy-Schwarz)

In fact, this inequality also holds in an inner product space.

Fact 1: Let V be an **inner product space**. Then the following **Cauchy-Schwarz inequality** holds:

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\| \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

Thus, given any inner product, we can use the quotient

$$\boxed{\cos\theta \underbrace{=}_{define} \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}}$$

to "define" the "angle" θ between the two vectors $\mathbf{v}, \mathbf{w}$ in the vector space V.

To see this, we know this ratio lies between -1 and 1, and defining the angle θ in this way makes sense.

Poll Question 1: Let V be an inner product space. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, which of the following is **NOT** true:

$A)$ $\langle 2\mathbf{u} + 7\mathbf{v},\ \mathbf{w} \rangle = 2\langle \mathbf{u},\ \mathbf{w} \rangle + 7\langle \mathbf{v},\ \mathbf{w} \rangle$
$B)$ $\langle 2\mathbf{u}, 2\mathbf{v} - 9\mathbf{w} \rangle = 4\langle \mathbf{u},\ \mathbf{v} \rangle - 18\langle \mathbf{u},\ \mathbf{w} \rangle$
$C)$ $\langle \mathbf{u}, \mathbf{v} \rangle = -\langle \mathbf{v},\ \mathbf{u} \rangle$