Lecture 17: Quick review from previous lecture

- A a vector space V equipped with a specific inner product is called an **inner product space**.
- $\langle \mathbf{v},\ \mathbf{w} \rangle$ is an **inner product** on V if the following hold:

  (1) Bilinearity:

$$\langle c\mathbf{u} + d\mathbf{v},\ \mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{w} \rangle + d\langle \mathbf{v},\ \mathbf{w} \rangle,$$
$$\langle \mathbf{u},\ c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u},\ \mathbf{v} \rangle + d\langle \mathbf{u},\ \mathbf{w} \rangle,$$

  (2) Symmetry: $\langle \mathbf{v},\ \mathbf{w} \rangle = \langle \mathbf{w},\ \mathbf{v} \rangle$,

  (3) Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$. Moreover, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

- Every inner product gives a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, that is, $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$

Today we will discuss

- Sec. 3.2 inequalities

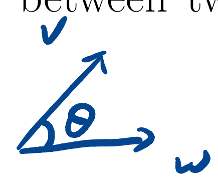
- Lecture will be recorded -

3.2 Inequalities

§ The Cauchy-Schwarz Inequality

Let's first recall the result in $\mathbb{R}^n$.

$\mathbb{R}^n$



In $\mathbb{R}^n$, we have seen from Calculus that the dot product between two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ can be geometrically characterized by

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta,$$

$-1 \leq \cos\theta \leq 1$

where θ is the angle between the two vectors. Thus,

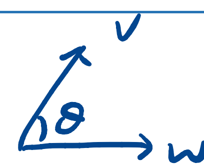
$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

In fact, this inequality also holds in an inner product space.

Fact 1: Let V be an **inner product space**. Then the following **Cauchy-Schwarz inequality** holds:

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\| \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

V (inner product space)



Thus, given any inner product, we can use the quotient

$$\boxed{\cos\theta \underbrace{=}_{define} \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}}$$

to "define" the "angle" θ between the two vectors $\mathbf{v}, \mathbf{w}$ in the vector space V.

To see this, we know this ratio lies between -1 and 1, and defining the angle θ in this way makes sense.

Example 1. Recall that $C^0 = C^0(I)$ denote the *vector space of continuous functions* on an interval I. We consider the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

Let $f(x) = \cos(x)$ and $g(x) = \sin(x)$.

(1) Consider $C^0[0, \pi]$, compute the inner product between f and g. Also compute their angle θ.

$$\langle f, g\rangle = \int_0^{\pi} f\, g\, dx = \int_0^{\pi} \cos x \sin x\, dx = \left.\frac{\sin^2 x}{2}\right|_0^{\pi} = 0.$$

$$\boxed{\cos\theta = \frac{\langle f, g\rangle}{\|f\|\,\|g\|}} = \frac{0}{\|f\|\,\|g\|} = 0.$$

$$\theta = \pi/2.$$

(2) Consider $C^0[0, \pi/2]$, compute the inner product between f and g. Also compute their angle θ.

$$\langle f, g\rangle = \int_0^{\pi/2} \cos x \sin x\, dx = \left.\frac{\sin^2 x}{2}\right|_0^{\pi/2} = \frac{1}{2}.$$

$$\|f\|^2 = \langle f, f\rangle = \int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \frac{1+\cos(2x)}{2}\, dx.$$

$$\Rightarrow \|f\| = \sqrt{\pi/4}$$

$$= \left.\frac{x}{2} + \frac{\sin(2x)}{4}\right|_0^{\pi/2} = \pi/4$$

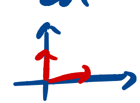
Similarly

$$\|g\|^2 = \int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2}\, dx = \pi/4. \Rightarrow \|g\| = \sqrt{\pi/4}$$

$$\boxed{\cos\theta = \frac{\langle f, g\rangle}{\|f\|\,\|g\|}} = \frac{\frac{1}{2}}{\pi/4} = \frac{2}{\pi}. \text{ Then } \theta = \cos^{-1}\left(\frac{2}{\pi}\right).\#$$

*Note that the choice of interval ($[0, \pi]$ versus $[0, \pi/2]$) changes the inner product between functions.

In $\mathbb{R}^2$, dot product



Definition: Let V be an inner product space. We call $\mathbf{x}, \mathbf{y} \in V$ are **orthogonal** (perpendicular) if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

In $C^0[0,\pi]$, $f \perp g$

In **Example 1(1)**, we saw that $f(x) = \cos(x)$ and $g(x) = \sin(x)$ satisfy

$$\langle f, g \rangle = \int_0^\pi f(x)g(x)dx = \int_0^\pi \cos(x)\sin(x)dx = 0$$

$C^0[0,\pi])$

Thus, we say f and g are **orthogonal** to each other on the interval $[0, \pi]$. However, they are NOT orthogonal on the interval $[0, \pi/2]$, see **Example 1(2)**.

$C^0([0,\pi/2])$

Fact 2: A vector $\mathbf{w}$ in $\mathbb{R}^n$ is orthogonal to vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ if and only if $\mathbf{w}$ is in the cokernel of the matrix $A = [\mathbf{v}_1, \dots, \mathbf{v}_k]$ (i.e. in the kernel of A^T)

Here we consider usual inner product (Dot product).

$$\boxed{\begin{aligned}&\operatorname{coker} A \\ &= \ker A^T\end{aligned}}$$

[To see this:]

$\langle v_1, w \rangle = 0, \dots, \langle v_k, w \rangle = 0.$

$$\Longleftrightarrow \begin{cases} v_1^T w = 0 \\ \vdots \\ v_k^T w = 0 \end{cases} \Longleftrightarrow \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix}_{k\times n} w = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Longleftrightarrow A^T w = \vec{0}$$

$\Longleftrightarrow w \in \underline{\operatorname{coker} A}.$

Example 2. Suppose we want to find a vector $\mathbf{w}$ that is orthogonal to $\mathbf{v}_1 = (1, 1, 0)^T$ and $\mathbf{v}_2 = (2, 1, -2)^T$ under usual inner product.

$\langle v_1, w \rangle = 0, \ \langle v_2, w \rangle = 0.$ Let $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$

$$\underbrace{\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}}_{A} w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ (homogeneous l. system)}$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \xrightarrow{②-2①} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -2 \end{pmatrix}.$$

② $b = -2c$

① $a = -b = 2c.$

$$w = \begin{pmatrix} 2c \\ -2c \\ c \end{pmatrix}, \quad \forall c \in \mathbb{R}.$$

[Proof of the Cauchy-Schwarz inequality:] $\left(|\langle x,y\rangle| \leq \|x\| \|y\| \right)$.

Suppose $y \neq 0$. we consider

$$\|x-cy\|^2 = \langle x-cy,\ x-cy\rangle.$$

$$\overset{(1)}{=} \langle x,x\rangle - c\langle x,y\rangle - c\langle y,x\rangle + c^2\langle y,y\rangle$$

$\langle x,y\rangle = \langle y,x\rangle$

$$\overset{(2)}{=} \|x\|^2 - 2c\langle x,y\rangle + c^2\|y\|^2.$$

Taking $\boxed{c = \frac{\langle x,y\rangle}{\|y\|^2}}$. Then

$$0 \leq \|x-cy\|^2 = \|x\|^2 - 2\frac{\langle x,y\rangle}{\|y\|^2}\langle x,y\rangle + \frac{\langle x,y\rangle^2}{\|y\|^{\cancel{4}\,2}}\cancel{\|y\|^2}$$

$$= \|x\|^2 - 2\frac{\langle x,y\rangle^2}{\|y\|^2} + \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$$= \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$$\Rightarrow \frac{\langle x,y\rangle^2}{\|y\|^2} \leq \|x\|^2 \Rightarrow \langle x,y\rangle^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x,y\rangle| \leq \|x\|\|y\|.$$

#

Q: In **Cauchy-Schwarz inequality** $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\|$, when does equality hold?

$v = cw$, for scalar $c \in \mathbb{R}$.

§ The Triangle Inequality

Fact 3: Let V be an inner product space. Then the following **Triangle Inequality** holds:

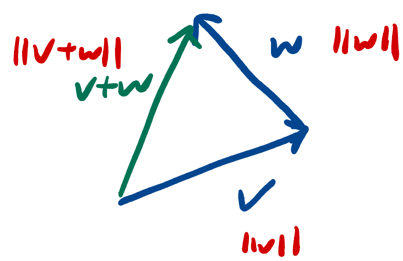
$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \ \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

* We skip the proof and refer it to the textbook.

[proof relies on Cauchy - Schwarz inequality and expanding $\|v+w\|^2$]

Remark: In the triangle inequality,

$$\text{the equality holds} \quad \text{if and only if} \quad \mathbf{v} = c\mathbf{w} \text{ and } c \geq 0$$



3.3 Norms

We have known that every inner product gives rise to a norm ($\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$) that can be used to measure the magnitude or length of the elements on an inner product space.

Let's recall that some properties satisfied by this norm, $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$, on an inner product space:

(1) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

(1) $\|cv\|^2 = \langle cv, cv \rangle = c^2 \langle v, v \rangle = c^2 \|v\|^2$

$\Rightarrow$ $\boxed{\|cv\| = |c| \, \|v\|}$

(2) $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = 0$.

(3) The triangle inequality:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

There are **other** natural measures of the "size" of a vector that satisfy these same three conditions, but that cannot be defined in terms of an inner product.

For example,
(1) on $\mathbb{R}^n$ we can measure the size of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ by the sum of absolute values of its entries:

$$\sum_{i=1}^{n} |v_i|$$

(2) on $C^0([a, b])$, we can measure the size of a continuous function f by the integral of its absolute value:

$$\int_a^b |f(x)| dx$$

Neither of these quantities can be defined in terms of an inner product; but they are still useful notions of the "size of vectors".

Definition: A **norm** on a vector space V assigns a non-negative real number $\|\mathbf{v}\|$ to each vector $\mathbf{v} \in V$, such that for every $\mathbf{v},\ \mathbf{w} \in V$ and $c \in \mathbb{R}$, the following axioms holds:

(1) **Positivity:** $\|\mathbf{v}\| \geq 0$; $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

(2) **Homogeneity:** $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

(3) **Triangle inequality:** $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

$\|x\| = \sqrt{\langle x, x \rangle}$

Remark: We checked that when "a norm is induced from an inner product, these three conditions are satisfied automatically". But in general, a norm need NOT arise from an inner product on V.

1. p **norm on** $\mathbb{R}^n$**.** We've already seen several on $\mathbb{R}^n$ and $C^0([a, b])$.

Example 1. We have seen that $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the norm induced from the dot product on $\mathbb{R}^n$.

Example 2. $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$. Let's check the three conditions to make sure this is a valid norm.

To be continued!