Lecture 18: Quick review from previous lecture

In an inner product space V,

- **Cauchy-Schwarz inequality** holds:

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\| \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

- **Triangle Inequality** holds:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\| \text{ for all } \mathbf{v}, \mathbf{w} \in V.$$

- We call $\mathbf{x}, \mathbf{y} \in V$ are **orthogonal (perpendicular)** if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- Every inner product gives a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, that is, $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$

Today we will discuss

- Sec. 3.3 norms

- Lecture will be recorded -

- The University provides free **peer tutor service**, which can be found in `https://www.lib.umn.edu/smart` (SMART Learning Commons)

Definition: A **norm** on a vector space V assigns a non-negative real number $\|\mathbf{v}\|$ to each vector $\mathbf{v} \in V$, such that for every $\mathbf{v},\ \mathbf{w} \in V$ and $c \in \mathbb{R}$, the following axioms holds:

(1) **Positivity:** $\|\mathbf{v}\| \geq 0$; $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$.

(2) **Homogeneity:** $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

(3) **Triangle inequality:** $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

$\|x\| = \sqrt{\langle x, x \rangle}$.

Remark: We checked that when "a norm is induced from an inner product, these three conditions are satisfied automatically". But in general, a norm need NOT arise from an inner product on V.

§ **1. p norm on $\mathbb{R}^n$.** We've already seen several on $\mathbb{R}^n$ and $C^0([a, b])$.

Example 1. We have seen that $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the norm ~~induced~~ from the usual inner product (dot product) on $\mathbb{R}^n$.

$= \sqrt{\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle}$ ..

2-norm

Example 2. $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$. Let's check the three conditions to make sure this is a valid norm.

1. Positivity: $\|x\|_1 = \sum_{i=1}^n |x_i| \geq 0$; $\|x\|_1 = 0 \iff |x_i| = 0, 1 \leq i \leq n$

$\iff x = 0$

2. Homogeneity: $c \in \mathbb{R}$, $\|cx\|_1 = \sum_{i=1}^n |cx_i| = |c| \left(\sum_{i=1}^n |x_i|\right) = |c| \, \|x\|_1$

3. Triangle inequality: $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_n)^T$

$\|x + y\|_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n (|x_i| + |y_i|) = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|$

$= \|x\|_1 + \|y\|_1$.

Then it's a norm. #

There is a generalization of these norms:

Definition: If $p \geq 1$, we define:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

This is often called the p **norm**.

Example 3. (1) Compute the 3 norm of $\mathbf{x} = (2, -1, 3)^T$.

$$\|x\|_3 = \left(|2|^3 + |-1|^3 + |3|^3\right)^{1/3}$$

$$= (8 + 1 + 27)^{1/3} = \underline{36^{1/3}} \; \#$$

(2) Compute the p norm of $\mathbf{x} = (1, \ldots, 1)^T$ in $\mathbb{R}^n$.

$$\|x\|_p = (1^p + \cdots + 1^p)^{1/p} = \underline{n^{1/p}} \; \#$$

§ **2.** L^p **norm on** $C^0[a, b]$ (the vector space of continuous functions).

Definition: We define the L^p norms on $C^0([a, b])$, for $p \geq 1$:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

Example 4. Compute the $3/2$ norm of $f(x) = x^2$ on $[-1, 1]$.

$$\|f\|_{3/2} = \left(\int_{-1}^{1} |f|^{3/2} dx\right)^{2/3} = \left(\int_{-1}^{1} |x^2|^{3/2} dx\right)^{2/3} = \left(\int_{-1}^{1} |x|^3 dx\right)^{2/3}$$

$$= \left(2\int_0^1 x^3 \, dx\right)^{2/3}$$

$$= \left(\frac{2}{4} x^4 \Big|_0^1\right)^{2/3} = \left(\frac{1}{2}\right)^{2/3} = \underline{2^{-2/3}} \; \#$$

$$\left(\frac{1}{2} = 2^{-1}\right)$$

§ 3. When $p = \infty$.

when $p = \infty$, we define ∞ norm on $\mathbb{R}^n$ by: $\mathbf{x} = (x_1, \ldots, x_n)$,

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

Similarly,

when $p = \infty$, we define L^∞ norm on $C^0([a,b])$ by:

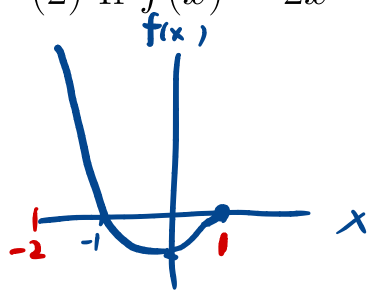
$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

Example 5. (1) If $\mathbf{x} = (1, 2, -7)^T$, then $\|\mathbf{x}\|_\infty =$ $\underline{7}$

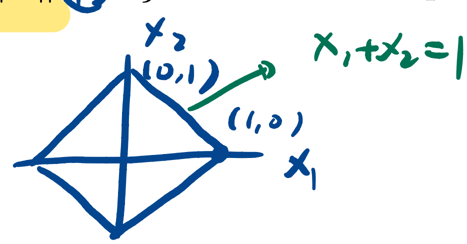
$|1|, |2|, |-7|$

(2) If $f(x) = 2x^2 - 2$, then on $[-2, 1]$, $\|f\|_\infty =$ $\underline{6}$

$|f(-2)|$



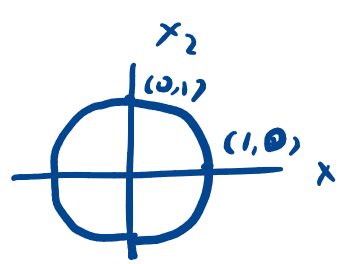
Example 6. Characterize the unit sphere $\{x \in \mathbb{R}^2 : \|x\| \le 1\}$ of $\mathbb{R}^2$ with respect to the following norms:



$\|\mathbf{x}\|_1 = \sum_{i=1}^2 |x_i|$

$S_1 = \{ x \mid \|x\|_1 = 1 \}$.

$|x_1| + |x_2| = 1$,

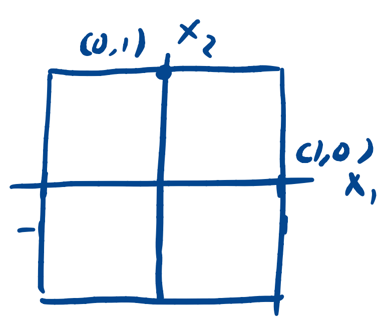


$\|\mathbf{x}\|_2 = (\sum_{i=1}^2 |x_i|^2)^{1/2}$

$S_2 = \{ x \mid \|x\|_2 = 1 \}$.

$\|\mathbf{x}\|_\infty = \max_{1 \le i \le 2} |x_i|$

$S_\infty = \{ x \mid \max\{|x_1|, |x_2|\} = 1 \}$.



§ 4. Matrix Norms.

If $\|\mathbf{v}\|$ is any norm on $\mathbb{R}^n$, it induces a "natural" norm on $\mathcal{M}_{n\times n}$, the vector space of n-by-n matrices.

Definition: The **matrix norm** (with respect to the norm $\|\cdot\|$ on $\mathbb{R}^n$) is defined as follows. If A is any n-by-n matrix, then

$$\|A\| = \max\{\|A\mathbf{u}\| : \|\mathbf{u}\| = 1\}$$

Proof. To show it is a norm, see pages 153–154 in the book for the proof. □

* In other words, $\|A\|$ is the maximum amount that A can change the norm of a unit vector $\mathbf{u}$ (one with $\|\mathbf{u}\| = 1$) when we apply A to $\mathbf{u}$.

The book calls $\|A\|$ above the **natural matrix norm** associated to the vector norm $\|\mathbf{v}\|$. It is also often called the **operator norm** of A.

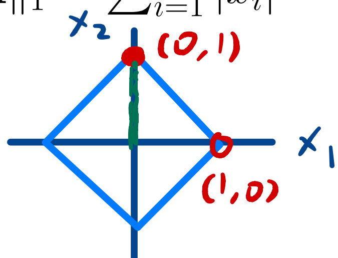
Example 7. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ which maps $\mathbb{R}^2$ to $\mathbb{R}^2$. Determine the matrix norm with respect to (w.r.t.) the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.

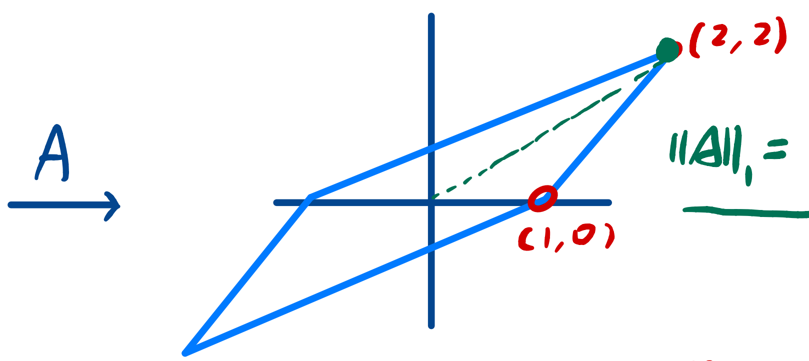
We depict the action of A on the unit balls of $\mathbb{R}^2$ w.r.t. norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.

- (1-norm) $\|\mathbf{x}\|_1 = \sum_{i=1}^{2} |x_i|$

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

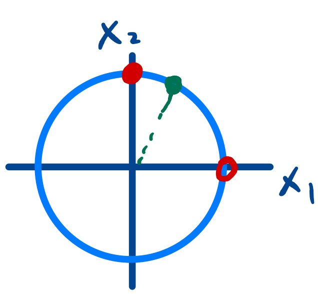
$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

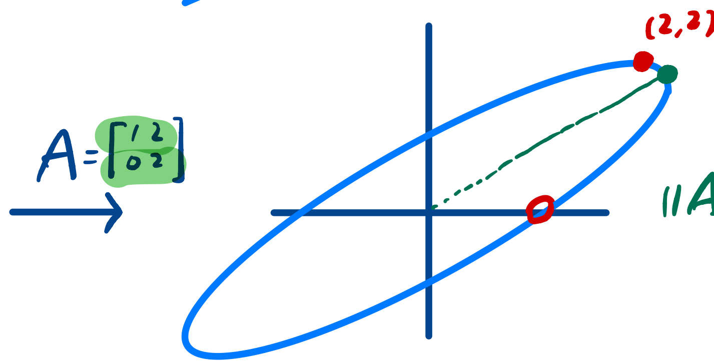




$\|A\|_1 = 4.$

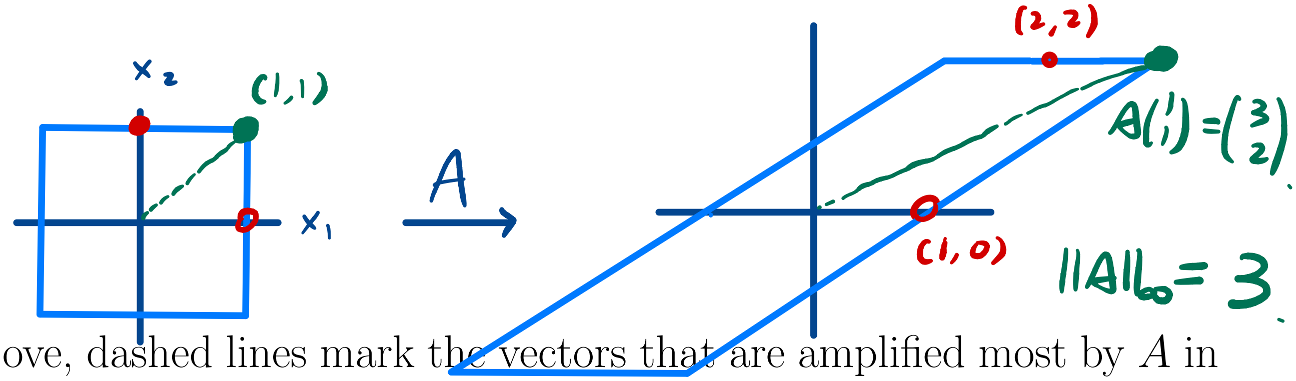
- (2-norm) $\|\mathbf{x}\|_2 = (\sum_{i=1}^{2} |x_i|^2)^{1/2}$





$\|A\|_2 \approx 2.9208$

- (∞-norm) $\|\mathbf{x}\|_\infty = \max_{1\le i\le 2} |x_i|$



*In figures above, dashed lines mark the vectors that are amplified most by A in each norm.

Fact 1:

$$(1)\quad \|A\mathbf{v}\| \le \|A\|\|\mathbf{v}\|, \qquad \text{for all } n\times n \text{ matrices } A,\ \mathbf{v}\in\mathbb{R}^n,$$

$$(2)\quad \|AB\| \le \|A\|\|B\|, \qquad \text{for all } n\times n \text{ matrices } A \text{ and } B.$$

[To see these:]

To be continued !

The 2^{nd} inequality implies that

Fact 2: If A is a square matrix and k is a positive integer, then

$$\|A^k\| \le \|A\|^k.$$

$k=3$: $\|A^3\| = \|A\underline{\underline{A^2}}\| \overset{\text{Fact 1(2)}}{\le} \|A\|\,\|A^2\|$

$\le \|A\|\,\|A\|\|A\| = \|A\|^3$

$k=4$: $\|A^4\| \le \|A\|^4$.

Example 8.

a) If $\|A\| = 3$ and $\|\mathbf{x}\| = 4$, what is the maximum possible value for $\|A\mathbf{x}\|$?

Fact 1.(1) $\|Ax\| \le \|A\| \|x\|$

$= 3 \cdot 4 = \underline{12}$ #.

b) Suppose that the $n \times n$ matrix A has norm $\|A\| = \lambda$, where λ is a scalar number satisfying $0 < \lambda < 1$. Find the limit of $\|A^k\|$ as k goes to ∞ (namely, find the value of $\lim_{k\to\infty} \|A^k\|$).

Fact 2

$\lim_{k\to\infty} \|A^k\| \le \lim_{k\to\infty} \|A\|^k = \lim_{k\to\infty} \lambda^k = \underline{0}$ #

since $0<\lambda<1$.

§ 5. Distance.

Every norms defines a **distance** between vector space elements, that is,

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

It satisfies

1. Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$
2. Positivity: $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \mathbf{v} = \mathbf{w}$.
3. Triangle inequality: $d(\mathbf{v}, \mathbf{w}) \le d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$