Lecture 19: Quick review from previous lecture

• If $p \geq 1$, we define the $p$ norm by

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

When $p = \infty$, we define $\infty$ norm by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

• If $p \geq 1$, we define the $L^p$ norm on $C^0([a, b])$ by

$$\|f\|_p = \left( \int_{a}^{b} |f(x)|^p dx \right)^{1/p}$$

When $p = \infty$, we define $L^\infty$ norm on $C^0([a, b])$ by

$$\|f\|_\infty = \max_{x \in [a, b]} |f(x)|$$

• $A$ is any $n$-by-$n$ matrix. We define the natural matrix norm (operator norm) of $A$ by

$$\|A\| = \max\{\|Au\| : \|u\| = 1\}$$

Today we will discuss

• Sec. 3.4 - 3.5 Positive Definite Matrices

- Lecture will be recorded -

• HW6 due today at 6pm.
Fact 1:

(1) $\|A\mathbf{v}\| \leq \|A\|\|\mathbf{v}\|$, for all $n \times n$ matrices $A$, $\mathbf{v} \in \mathbb{R}^n$;

(2) $\|AB\| \leq \|A\|\|B\|$, for all $n \times n$ matrices $A$ and $B$.

[To see these:

(1) $\mathbf{v} \neq \mathbf{0}$. Let $w = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ (unit vector) ($\|w\| = 1$).

Then $\|Aw\| = \max \{\|Au\| \mid \|u\| = 1\} = \|A\|w$.

(2) Taking $u$ with $\|u\| = 1$.

$\|ABu\| = \|A(Bu)\| \leq \|A\|\|Bu\| \leq \|A\|\|B\|\|u\| \leq \|A\|\|B\|\|w\| = \|A\|\|B\|\|w\|.$

$\max \{\|ABu\| \mid \|u\| = 1\} \leq \|A\|\|B\|.$

$\|AB\|$]

§ 5. Distance.

Every norms defines a distance between vector space elements, that is,

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$ 

It satisfies

1. Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$

2. Positivity: $d(\mathbf{v}, \mathbf{w}) = 0 \iff \mathbf{v} = \mathbf{w}$.

3. Triangle inequality: $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$.
Example 9. Suppose $A$ is a $n$-by-$n$ matrix. Suppose that $x = (x_1, \ldots, x_n)$ is a vector in $\mathbb{R}^n$. Recall that the 2 norm is defined as

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$ 

Find the operator norm of $A = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}$. That is, find $\|A\|_2 = \max\{\|Au\|_2 : \|u\|_2 = 1\}$. Let $u = (a, b)^T$ with $\|u\|_2 = 1$. $a^2 + b^2 = 1$.

$$Au = \begin{pmatrix} 3a \\ -5b \end{pmatrix}, \quad \|Au\|_2^2 = (3a)^2 + (-5b)^2 = 9a^2 + 25b^2 = 9a^2 + 25(1-a^2) = -16a^2 + 25.$$ 

Find $\max \|Au\|_2$. (is the same as find $\max f(a)$)

$$f(a) = -16a^2 + 25, \quad -1 \leq a \leq 1.$$ 

Critical pt: $f'(a) = 0 \Rightarrow -32a = 0 \Rightarrow a = 0$ 

$f(0) = 25$ (✓)

End pts: $f(1) = 9$ 

$f(-1) = 9$. Then $\max \|Au\|_2 = \sqrt{25} = 5$

**Fact:** $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_n \end{bmatrix}$, then $\|D\|_2 = \max\{1 \|d_j\|_1 \}$. 

$A(0) = (0, -5)$ vector amplified most by $A$. 

$A(1) = (3, 0)$ 

$A(0) = (0, -5)$
We have seen that the following inner products on $\mathbb{R}^n$:

- standard inner product $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n = x^TIy$
- For $c_1 > 0, \cdots, c_n > 0$, $\langle x, y \rangle = c_1x_1y_1 + \cdots + c_nx_ny_n = x^TDy$ where $D = \text{diag}(c_1, \ldots, c_n)$.
- For nonsingular matrix $A$, $\langle x, y \rangle = x^TA^TAy$ is also an inner product.

They all have been of the following form:

$$\langle x, y \rangle = x^T \begin{bmatrix} K \end{bmatrix}_{\text{symmetric}} y$$

for some symmetric matrix $K$.

**Q:** Are there any other types of inner products on $\mathbb{R}^n$?

**Ans:** In fact, all inner products on $\mathbb{R}^n$ are of the form $\langle x, y \rangle = x^TKy$, for some symmetric matrix $K$.

However, it is not true that any symmetric matrix $K$ can define an inner product! Only a special type of matrix $K$ can do this, that is,

**Definition:** An $n \times n$ matrix $K$ is called **positive definite** if

1) it is symmetric, $K^T = K$, and satisfies
2) the positivity condition

$$x^TKx > 0 \quad \text{for all } 0 \neq x \in \mathbb{R}^n.$$ 

We write “$K > 0$” to mean that $K$ is positive definite matrix.

**Warning:** The condition $K > 0$ does NOT mean that all the entries of $K$ are positive.
Fact 1: Every inner product on \( \mathbb{R}^n \) is given by
\[
\langle x, y \rangle = x^T K y
\]
for all \( x, y \in \mathbb{R}^n \),
where \( K \) is a **symmetric, positive definite** \( n \times n \) matrix.

Definition: For a matrix \( K \), the function
\[
q(x) = x^T K x
\]
is called a **quadratic form**.
Moreover, quadratic form is called **positive-definite** if \( q(x) > 0 \) for all \( x \neq 0 \).

Example 1. \( A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \) is positive definite. We denote it as \( A > 0 \).

1. \( A = A^T \)
2. \( q(x) = x^T A x = (x_1, x_2) \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 5x_2^2 > 0 \) if \( x \neq 0 \).

Example 2. Check if \( K = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \) is positive definite.

1. \( K = K^T \)
2. \( q(x) = x^T K x = (x_1, x_2) \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} 
= 2x_1^2 - 2x_1x_2 + 5x_2^2 
= (x_1 - x_2)^2 + x_1^2 + 4x_2^2 
> 0 \) if \( x \neq 0 \).

Then \( K > 0 \) (positive definite).
Example 3. Check if

\[ A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \]

is positive definite.

1. \( A = A^T \)
2. \( g(x) = x^T A x = x_1^2 + 6x_1x_2 + 2x_2^2 \).
   Taking \( x = (1, -1) \). \( g(1, -1) = 1 - 6 + 2 < 0 \).

Warning: positive definite matrices may have negative entries, while matrices with all positive entries may not always be positive definite.

§ The positive definite 2 × 2 matrices.

Q: How can we tell if a matrix \( K \) is positive definite? We obviously can’t evaluate \( x^T K x \) for all vectors \( x \) every time!

Fact 2: Any symmetric 2-by-2 matrix \( A \):

\[ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]

is positive definite if and only if

\[ a > 0 \quad \text{and} \quad ac - b^2 > 0 \]

Revisit Example 2. \( K = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \)

\( a_{11} > 0 \)
\( \det K = 2 \cdot 5 - (-1)^2 > 0 \).

Then \( K > 0 \).
§ The positive definite \( n \times n \) (any square size) matrices.

Q: Is there a simple characterization for positive definite matrices of any size?

Recall (page 45-47): Any regular symmetric matrix \( A \) can be factored in the form

\[
A = LDL^T
\]

where \( L \) is lower unitriangular and \( D \) is diagonal.

(*This factorization is computed via “adding/subtracting row to/from other row” only, no permuting row operation.)

Fact 3: An \( n \)-by-\( n \) matrix \( A \) is positive definite if and only if it is:

1. symmetric;
2. regular, hence \( A = LDL^T \); and
3. \( D \) has all positive diagonal entries (i.e. \( A \) has positive pivots).

It immediately implies that

Fact 4: If a matrix \( A \) is positive definite, then it is nonsingular.

[To see this: Since \( A \) has only positive pivots, \( \det A \neq 0 \) so \( A \) is nonsingular. ]

Example 4. Determine if the following matrix is positive definite:

\[
a) \quad A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 5 \end{pmatrix}
\]

Then \[
D = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix}
\]

So \( A \) is \underline{not} positive definite.

\[
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\]
b) \[ A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix} \]

\[ \Rightarrow \] Then \( A > 0 \).

**Definition:** If a matrix \( A \) satisfies \( x^T A x \geq 0 \) for all vectors \( x \), it is called positive semidefinite.

**Remark:** Every positive definite matrix is also positive semidefinite; but the converse is not true:

\[ \text{positive definite} \Rightarrow \text{positive semidefinite} \]

since a positive semidefinite matrix \( A \) might have \( x^T A x = 0 \) for \( x \neq 0 \).

**Example 5.** The matrix \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \) is positive semidefinite, but not positive definite.

To be continued!
Definitions:

- A matrix $A$ is **negative definite** if $x^T A x < 0$ for all $x \neq 0$.
- Similarly, a matrix $A$ is **negative semidefinite** if $x^T A x \leq 0$ for all $x$.
- If a matrix is neither positive or negative semidefinite, it is called **indefinite**. This means that there are vectors $x$ and $y$ with $x^T A x > 0$ and $y^T A y < 0$.

*Only “positive definite” matrices define inner products, via $\langle x, y \rangle = x^T A y$. 
Poll Question 1: The norm of a vector $\mathbf{x}$ in a vector space is always nonnegative (that is, $\|\mathbf{x}\| \geq 0$).

A) Yes
B) No

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