Lecture 19: Quick review from previous lecture

- If $p \geq 1$, we define the p **norm** by
$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$
When $p = \infty$, we define ∞ norm by
$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$
- If $p \geq 1$, we define the L^p **norm** on $C^0([a,b])$ by
$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$
When $p = \infty$, we define L^∞ norm on $C^0([a,b])$ by
$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$
- A is any n-by-n matrix. We define the **natural matrix norm** (**operator norm**) of A by
$$\|A\| = \max\{\|A\mathbf{u}\| : \|\mathbf{u}\| = 1\}$$

Today we will discuss

- Sec. 3.4 - 3.5 Positive Definite Matrices

- Lecture will be recorded -

- HW6 due today at 6pm.

Fact 1:

$$(1) \quad \|A\mathbf{v}\| \leq \|A\|\|\mathbf{v}\|, \qquad \text{for all } n \times n \text{ matrices } A,\ \mathbf{v} \in \mathbb{R}^n,$$

$$(2) \quad \|AB\| \leq \|A\|\|B\|, \qquad \text{for all } n \times n \text{ matrices } A \text{ and } B.$$

[To see these:] (1) $v \neq 0$. Let $w = \frac{v}{\|v\|}$ (unit vector) $(\|w\|=1)$.

$\|Aw\| \leq \max\{\|Au\| \mid \|u\|=1\} = \|A\|$.

Then $\|Aw\| \leq \|A\|$.

$\|A\frac{v}{\|v\|}\| \leq \|A\|$.

$\frac{1}{\|v\|}\|Av\| \leq \|A\|$. implies $\|Av\| \leq \|A\|\|v\|$ #

(2) Taking u with $\|u\|=1$.

$\|ABu\| = \|A(Bu)\| \overset{(1)}{\leq} \|A\|\|Bu\| \overset{(1)}{\leq} \|A\|\|B\|\|u\|$.

$\overset{\|u\|=1}{=} \|A\|\|B\|$

$\underbrace{\max\{\|ABu\| \mid \|u\|=1\}}_{\|AB\|} \leq \|A\|\|B\|$.

§ 5. Distance.

Every norms defines a **distance** between vector space elements, that is,

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

It satisfies

1. Symmetry: $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$
2. Positivity: $d(\mathbf{v}, \mathbf{w}) = 0 \Leftrightarrow \mathbf{v} = \mathbf{w}$.
3. Triangle inequality: $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{z}) + d(\mathbf{z}, \mathbf{w})$

Example 9. Suppose A is a n-by-n matrix. Suppose that $\mathbf{x} = (x_1, \cdots, x_n)$ is a vector in $\mathbb{R}^n$. Recall that the 2 norm is defined as

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Find the operator norm of $A = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}$. That is, find $\|A\|_2 = \max\{\|A\mathbf{u}\|_2 : \|\mathbf{u}\|_2 = 1\}$. Let $u = (a, b)^T$ with $\|u\|_2 = 1$. $a^2 + b^2 = 1$.

$-1 \le a, b \le 1$.

$$Au = \begin{pmatrix} 3a \\ -5b \end{pmatrix}. \quad \|Au\|^2 = (3a)^2 + (-5b)^2$$
$$= 9a^2 + 25b^2$$
$$= 9a^2 + 25(1 - a^2)$$
$$= -16a^2 + 25.$$

Find $\max \|Au\|_2$. (is the same as find $\max f(a)$)

$f(a) = -16a^2 + 25, \quad -1 \le a \le 1.$

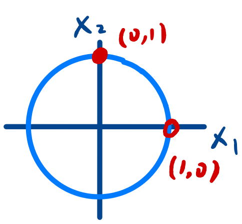
Critical pt: $f'(a) = 0 \Rightarrow -32a = 0 \Rightarrow a = 0$

$f(0) = 25$ (✓)

End pts: $f(1) = 9$

$f(-1) = 9$.

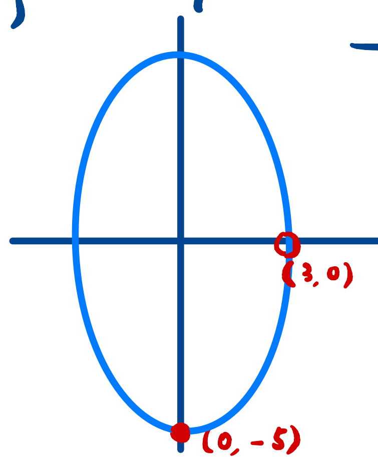
Then $\max \|Au\|_2 = \sqrt{25} = 5$



$A = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$ $\longrightarrow$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$



$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$

vector amplified most by A.

$\boxed{\text{Fact:}}$ $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$, $\|D\|_2 = \max_{1 \le j \le n} \{|d_j|\}$

3.4 - 3.5 Positive Definite Matrices

We have seen that the following inner products on $\mathbb{R}^n$:

- standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n = \mathbf{x}^T I \mathbf{y}$
- For $c_1 > 0, \cdots, c_n > 0$, $\langle \mathbf{x}, \mathbf{y} \rangle = c_1 x_1 y_1 + \ldots + c_n x_n y_n = \mathbf{x}^T D \mathbf{y}$ where $D = \operatorname{diag}(c_1, \ldots, c_n)$.
- For nonsingular matrix A, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A^T A \mathbf{y}$ is also an inner product

$(A^T A)^T = A^T A$

They all have been of the following form:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \underbrace{K}_{symmetric} \mathbf{y}$$

for some **symmetric** matrix K.

Q: Are there any other types of inner products on $\mathbb{R}^n$?
Ans: In fact, all inner products on $\mathbb{R}^n$ are of the form $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$, for some symmetric matrix K.

However, it is not true that any symmetric matrix K can define an inner product! Only a special type of matrix K can do this, that is,

Definition: An $n \times n$ matrix K is called **positive definite** if
1) it is symmetric, $K^T = K$, and satisfies
2) the positivity condition

$$\mathbf{x}^T K \mathbf{x} > 0 \qquad \text{for all } \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n.$$

We write "$K > 0$" to mean that K is positive definite matrix.

Warning: The condition $K > 0$ does NOT mean that all the entries of K are positive.

Fact 1: Every inner product on $\mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where K is a **symmetric, positive definite** $n \times n$ matrix.

Definition: For a matrix K, the function

$$q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x}$$

is called a **quadratic form**.
Moreover, quadratic form is called **positive-definite** if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

Example 1. $A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ is positive definite. We denote it as $A > 0$.

① $A = A^T$

② $q(x) = x^T A x = (x_1 \ x_2) \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 5x_2^2$
> 0 if $x \neq 0$.

Then $\underline{A > 0}$

Example 2. Check if

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$$

is positive definite.

① $K = K^T$

② $q(x) = x^T K x = (x_1 \ x_2) \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= 2x_1^2 - 2x_1x_2 + 5x_2^2$.

$= (x_1 - x_2)^2 + x_1^2 + 4x_2^2$

> 0 if $x \neq 0$

Then $\underline{K > 0}$ (positive definite)

Example 3. Check if

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}.$$

is (NO) positive definite. ① $A = A^T$

② $q(x) = x^T A x = x_1^2 + 6x_1 x_2 + 2x_2^2$.

Taking $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $q\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 6 + 2 < 0$.

Ex 2

Warning: positive definite matrices may have negative entries, while matrices with all positive entries may not always be positive definite.

Ex 3

§ The positive definite 2×2 matrices.

Q: How can we tell if a matrix K is positive definite? We obviously can't evaluate $\mathbf{x}^T K \mathbf{x}$ for all vectors $\mathbf{x}$ every time!

> **Fact 2:** Any symmetric 2-by-2 matrix A:
>
> $$A = \begin{pmatrix} \boxed{a} & b \\ b & c \end{pmatrix}$$
>
> is **positive definite** if and only if
>
> $$a > 0 \quad \text{and} \quad \overbrace{ac - b^2}^{\det A} > 0$$

Revisit **Example 2.** $K = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$

$a_{11} > 0$

$\det K = 2 \cdot 5 - (-1)^2 > 0$.

Then $K > 0$.

§ The positive definite $n \times n$ (any square size) matrices.

Q: Is there a simple characterization for positive definite matrices of any size?

all pivots $\neq 0$
no permutation

Recall (page 45-47): Any regular symmetric matrix A can be factored in the form

$$\boxed{A = LDL^T}$$

where L is lower unitriangular and D is diagonal.
(*This factorization is computed via "adding/subtracting row to/from other row" only, no permuting row operation.)

Fact 3: An n-by-n matrix A is **positive definite** if and only if it is:
(1) **symmetric**;
(2) **regular**, hence $A = LDL^T$; and
(3) D has all **positive** diagonal entries (i.e. A has positive pivots).

It immediately implies that

Fact 4: If a matrix A is positive definite, then it is nonsingular.

[To see this: Since A has only positive pivots, $\det A \neq 0$ so A is nonsingular.]

$= \det D$

Example 4. Determine if the following matrix is positive definite:

a) $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 5 \end{pmatrix} \xrightarrow[3-2(1)]{2+1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{3+2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix} = U.$

Then $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

So A is NOT positive definite.

.

$$\text{b)}\quad A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{pmatrix} \xrightarrow[③+①]{②-2①} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{pmatrix} = U.$$

U has all positive pivots.

Then $A > 0$. #.

Definition: If a matrix A satisfies $\mathbf{x}^T A\mathbf{x} \geq 0$ for all vectors $\mathbf{x}$, it is called **positive semidefinite**.

Remark: Every positive definite matrix is also positive semidefinite; but the converse is not true:

$$\boxed{\text{positive definite} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{positive semidefinite}}$$

since a positive semidefinite matrix A might have $\mathbf{x}^T A\mathbf{x} = 0$ for $\mathbf{x} \neq \mathbf{0}$.

Example 5. The matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite, but not positive definite.

To be continued!

Definitions:

- a matrix A is **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- Similarly, a matrix A is **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x}$.
- If a matrix is neither positive or negative semidefinite, it is called **indefinite**. This means that there are vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{y}^T A\mathbf{y} < 0$.

*Only "positive definite" matrices define inner products, via $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}$.

Poll Question 1: The norm of a vector $\mathbf{x}$ in a vector space is always nonnegative (that is, $\|\mathbf{x}\| \geq 0$).

A) Yes
B) No

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