Lecture 2: Quick review from previous lecture

• **Gaussian elimination** to solve a linear system \( Ax = b \).

• Matrix and basic operations, including addition, multiplication....

• **Zero matrix** is denoted by \( O \) or \( O_{m \times n} \).

• \( I_n \) is the \( n \text{-by-} n \) identity matrix and can be represented as \( I_n = \text{diag}(1, \cdots, 1) \).

\[
I_n = \begin{pmatrix}
1 & \cdots & 0 \\
0 & \cdots & 1
\end{pmatrix}_{n \times n} = \text{diag}(1, 1, \cdots, 1).
\]

Recall \( \text{Ex: diag } (a_1, a_2, a_3) = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix} \).

Today we will

• continue discuss Sec. 1.2. Matrices and Vectors and Basic Operations

• discuss Sec. 1.3 Gaussian Elimination

- Lecture will be recorded -

• The first problem set has been posted on Canvas. It is due next Friday (1/29) at 6pm.
Example 2: Solve the system
\[
\begin{align*}
  x + 2y + 2z &= 2 \\
  2x + 6y &= 1 \\
  4x + 4z &= 20
\end{align*}
\]

Augmented matrix
\[
\begin{pmatrix}
  1 & 2 & 2 & | & 2 \\
  2 & 6 & 0 & | & 1 \\
  4 & 0 & 4 & | & 20
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 2 & 2 & | & 2 \\
  0 & -4 & -3 & | & -3 \\
  0 & -8 & -4 & | & 12
\end{pmatrix}
\]

Use \( \text{①} \) to eliminate "\( x \)" from \( \text{③} \),

\[
\begin{pmatrix}
  1 & 2 & 2 & | & 2 \\
  0 & -4 & -3 & | & -3 \\
  0 & 0 & -20 & | & 0
\end{pmatrix}
\]

To solve \( x, y, z \),
same as right-hand side.

To solve \( x, y, z \) by using back substitution,

\[
\begin{align*}
  \text{③} &: \quad z = 0. \\
  \text{②} &: \quad 2y - 4(0) = -3 \quad \Rightarrow \quad y = -\frac{3}{2} \\
  \text{①} &: \quad x + 2(-\frac{3}{2}) + 0 = 2 \quad \Rightarrow \quad x = 5.
\end{align*}
\]

This is called "Gaussian elimination process."
1.3 Gaussian Elimination

In Gaussian elimination process, when we reach the \( j \)th row, element \((j, j)\) of the new augmented matrix is called the **pivot** for that row.

We look at the example:

**Example 1:** Find pivots of the system:

\[
\begin{align*}
    x + 2y + 2z &= 2 \\
    2x + 10y &= 1 \\
    4x + 20y + 4z &= 0
\end{align*}
\]

\[
\text{augmented matrix} \\
\begin{pmatrix}
    1 & 2 & 2 \\
    2 & 10 & 0 \\
    4 & 20 & 4
\end{pmatrix}
\]

\[
\begin{align*}
    &\rightarrow (2) - 2(1) \\
    &\rightarrow (3) - 4(1)
\end{align*}
\]

\[
\begin{pmatrix}
    1 & 2 & 2 \\
    0 & 6 & -4 \\
    0 & 12 & -8
\end{pmatrix}
\]

1st pivot

2nd pivot

3rd pivot

**Exercise:** To solve \( x, y, z \),
If at any point in the process one of the pivots is 0, then we are stuck! We can’t use a row with a zero pivot to eliminate the entries beneath that pivot.

**Example 2:** Suppose we are solving a 4-by-4 system and after using the first row to eliminate entries (2, 1), (3, 1), and (4, 1), we have the following matrix:

\[
\begin{pmatrix}
5 & 2 & 3 & 5 & 2 \\
0 & 0 & 2 & 6 & 9 \\
0 & 1 & 3 & 8 & 3 \\
0 & 2 & 5 & 1 & 8 \\
\end{pmatrix}
\]

• How to fix this?

We permute row 2 with another row (will discuss more later).

**Definition:** If a matrix $A$ has all non-zero pivots, then this matrix $A$ is called **regular**.

That is, regular matrices are those for which Gaussian elimination can be performed without switching the order of rows.

For instance, the matrix in **Example 1** is regular since all its pivots are NOT zero.
Remark:

- **Adding/subtracting** a multiple of one row to/from another row is called an **elementary row operation**.
- Each elementary row operation is associated with an **elementary matrix**, defined by applying the elementary row operation to the identity matrix.

**Example 3.** The $3 \times 3$ **elementary matrix** associated with adding 3 times the $3^{rd}$ row to the $1^{st}$ row is:

$$I_3 \rightarrow \begin{pmatrix} 0 & +3 & 3 \end{pmatrix} \hspace{1cm} E = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ elementary matrix}$$

- Multiplying a matrix $A$ on the left by an elementary matrix $E$ performs the associated row operation on $A$. For example, check that:

$$EA = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a + 3g & b + 3h & c + 3i \\ d & e & f \\ g & h & i \end{pmatrix}$$

**Properties about elementary matrix**

- Suppose $E$ is a 3-by-3 **elementary matrix** that adds 7 times the $1^{st}$ row to the $3^{rd}$ row. Then:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$
Then

\[
EA = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
7 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix} = \begin{pmatrix}
a & b & c \\
d & e & f \\
g + 7a & h + 7b & i + 7c
\end{pmatrix}
\]

- How to **UNDO** the effect of this row operation? (To get original A)

\[
E^{-1} = \begin{pmatrix}
0 & 0 & 0 \\
-7 & 0 & 1
\end{pmatrix}
\]

\[
E^{-1}E = I_3, \quad \text{also} \quad EE^{-1} = I_3
\]

\[
\overset{\text{undo}}{E^{-1}EA} = A.
\]

§ Some observations of \(E_1^{-1}E_2^{-1} \ldots E_m^{-1}\), where \(E_i\) is elementary matrix with lower triangular form: We first consider \(m = 3\). Let

\[
E_1 = \begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
b & 0 & 1
\end{pmatrix}, \quad E_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & c & 1
\end{pmatrix}
\]

\[
(\text{undo}) \quad E_1^{-1} = \begin{pmatrix}
-1 & 0 & 0 \\
-a & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix}
10 & 0 & 0 \\
0 & 1 & 0 \\
-b & 0 & 1
\end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix}
10 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

Then

\[
E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
-a & 1 & 0 \\
-b & -c & 1
\end{pmatrix}, \quad \overset{\text{lower triangular form}}{\text{meaning zero above diagonal}}
\]

In general, \(E_1^{-1} \ldots E_m^{-1}\) has the form

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & c & 1
\end{pmatrix}, \quad \overset{\text{lower triangular}}{\text{meaning zero above diagonal}}
\]
In particular, $E_1^{-1}E_2^{-1}E_3^{-1}E_3E_2E_1 = I$ (identity matrix) which also implies

undo effects

Thus, we observe the following fact in the Gaussian elimination process:

**Summary:** For any **regular** matrix $A$, we can multiply it on the left by a sequence of elementary matrices $E_1, \ldots, E_m$, so that the product is an upper triangular matrix $U$, namely:

$$E_mE_{m-1} \cdots E_1A = U$$

Then

$$\begin{pmatrix} E_1^{-1} & \cdots & E_m^{-1} \end{pmatrix} E_mE_{m-1} \cdots E_1A = \begin{pmatrix} E_1^{-1} & \cdots & E_m^{-1} \end{pmatrix} U.$$

**Fact 1:**

(1) We have shown that any **regular** matrix $A$ can be factored as

$$A = LU,$$

where $U$ is upper triangular and $L$ is lower triangular.

Furthermore, $L$ has 1’s on its main diagonal, and $U$ has non-zero elements on its main diagonal (the pivots of $A$).

(2) $L, \tilde{L}$ are $n \times n$ lower triangular matrices, so is $L\tilde{L}$.

(3) $U, \tilde{U}$ are $n \times n$ upper triangular matrices, so is $U\tilde{U}$.
Example 4: Find $LU$ factorization of the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix}, \quad L = I_3.$$ 

$$E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & 4 & -2 \end{pmatrix}, \quad L = E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \left( \begin{array}{c} \square \\ \end{array} \right) = U$$

To be continued!
Poll Question: Which of the following is “linear” system?

A)
\[
\begin{align*}
    x + 2y + 2z &= 2 \\
    10y - z &= 1 \\
    4x + 4z^2 &= 0
\end{align*}
\]

B)
\[
\begin{align*}
    x + 2^4y + 3z &= 1 \\
    2x + 10y &= -2 \\
    4x + 11^3y &= 1
\end{align*}
\]

Caution: After clicking submit, you will NOT be able to resubmit your answer!

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.