Lecture 2: Quick review from previous lecture

- **Gaussian elimination** to solve a linear system $A\mathbf{x} = \mathbf{b}$.
- Matrix and basic operations, including addition, multiplication....
- **Zero matrix** is denoted by O or $O_{m\times n}$.
- I_n is the n-by-n **identity matrix** and can be represented as $I_n = \operatorname{diag}(1, \cdots, 1)$.

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n\times n} = \operatorname{diag}(1, 1, \cdots, 1).$$

Recall <u>Ex</u>: $\operatorname{diag}(a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$

Today we will

- continue discuss Sec. 1.2. Matrices and Vectors and Basic Operations
- discuss Sec. 1.3 Gaussian Elimination

- Lecture will be recorded -

- The first problem set has been posted on Canvas. It is due next Friday (1/29) at 6pm.

Example 2: Solve the system $\begin{cases} x+2y+2z &= 2 \\ 2x+6y &= 1 \\ 4x+4z &= 20 \end{cases}$

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 6 & 0 & 1 \\ 4 & 0 & 4 & 20 \end{array}\right)$$

$$\begin{cases} 1x+2y+2z &= 2 \quad - ① \\ 2x+6y &= 1 \quad - ② \\ 4x+4z &= 20 \quad - ③ \end{cases}$$

Use ① to eliminate "x" from ② ③.

$$\xrightarrow[③-4①]{②-2①} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 2 & -4 & -3 \\ 0 & -8 & -4 & 12 \end{array}\right)$$

$$\xrightarrow[③-4①]{②-2①} \begin{cases} x+2y+2z=2 \\ 2y-4z=-3 \\ -8y-4z=12 \end{cases}$$ New system.

Use ② to eliminate "y" from ③

$$\xrightarrow{③+4②} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 2 & -4 & -3 \\ 0 & 0 & -20 & 0 \end{array}\right)$$

$$\xrightarrow{③+4②} \begin{cases} x+2y+2z=2 & -① \\ 2y-4z=-3 & -② \\ -20z=0. & -③ \end{cases}$$

To solve x, y, z, same as right-hand side.

This is called "Gaussian elimination process".

To solve x, y, z by using back substitution.

③ : $z=0$.

② : $2y-4(0)=-3$

$y=-3/2$.

① : $x+2\left(-\frac{3}{2}\right)+0=2$.

$x=5$.

$(5, -3/2, 0)$. #

1.3 Gaussian Elimination

In Gaussian elimination process, when we reach the j^{th} row, element (j, j) of the new augmented matrix is called the **pivot** for that row.

We look at the example:

Example 1: Find pivots of the system:

$$\begin{cases} x + 2y + 2z &= 2 \\ 2x + 10y &= 1 \\ 4x + 20y + 4z &= 0 \end{cases}$$

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 10 & 0 & 1 \\ 4 & 20 & 4 & 0 \end{array}\right) \xrightarrow[③-4①]{②-2①} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 6 & -4 & -3 \\ 0 & 12 & -4 & -8 \end{array}\right)$$

1st pivot

$$\xrightarrow{③-2②} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 6 & -4 & -3 \\ 0 & 0 & 4 & -2 \end{array}\right)$$

2nd pivot

3rd pivot.

Exercise: To solve x, y, z,

✓ If at any point in the process one of the pivots is 0, then we are stuck! We can't use a row with a zero pivot to eliminate the entries beneath that pivot.

Example 2: Suppose we are solving a 4-by-4 system and after using the first row to eliminate entries $(2,1)$, $(3,1)$, and $(4,1)$, we have the following matrix:

$$\left(\begin{array}{cccc|c} 5 & 2 & 3 & 5 & 2 \\ 0 & 0 & 2 & 6 & 9 \\ 0 & 1 & 3 & 8 & 3 \\ 0 & 2 & 5 & 1 & 8 \end{array}\right)$$

0 pivot

- How to fix this? We permute row ② with other row (will discuss more later).

Definition: If a matrix A has all non-zero pivots, then this matrix A is called **regular**.

That is, regular matrices are those for which Gaussian elimination can be performed without switching the order of rows.

For instance, the matrix in **Example 1** is regular since all its pivots are NOT zero.

Remark:

- Adding/subtracting a multiple of one row to/from another row is called an **elementary row operation**.
- Each elementary row operation is associated with an **elementary matrix**, defined by applying the elementary row operation to **the identity matrix**.

Example 3. The 3×3 **elementary matrix** associated with adding 3 times the 3^{rd} row to the 1^{st} row is:

$$I_3 \xrightarrow{①+3③} E = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ elementary matrix.}$$

- Multiplying a matrix A on the left by an elementary matrix E performs the associated row operation on A. For example, check that:

$$EA = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a + 3g & b + 3h & c + 3i \\ d & e & f \\ g & h & i \end{pmatrix}$$

§ Properties about elementary matrix

- Suppose E is a 3-by-3 elementary matrix that adds 7 times the 1^{st} row to the 3^{rd} row. Then:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

Then

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g+7a & h+7b & i+7c \end{pmatrix}$$

- How to UNDO the effect of this row operation? (To get original A)

" substracting 7 times 1st row from 3rd row"

elementary matrix

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}.$$

$$\underbrace{E^{-1}}_{\text{undo}} E = I_3. \quad \text{also} \quad E E^{-1} = I_3$$

$$E^{-1}EA = A.$$

§ **Some observations of $E_1^{-1}E_2^{-1}\ldots E_m^{-1}$, where E_i is elementary matrix with lower triangular form:** We first consider $m = 3$. Let

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$$

(undo) $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix}$.

Then $E_1^{-1}E_2^{-1}E_3^{-1} = \underline{\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & -c & 1 \end{pmatrix}}$. #

lower triangular form

meaning zero above diagonal

In general, $E_1^{-1} \cdots E_m^{-1}$ has the form

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & & 0 \\ * & * & \ddots & \vdots \\ * & * & * & 1 \end{pmatrix}, \text{ lower triangular.}$$

In particular, $\underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{\text{undo}}E_3E_2E_1 = I$ (identity matrix) which also implies

$$\boxed{\underbrace{\boxed{E_1^{-1}E_2^{-1}E_3^{-1}}}_{\text{undo effects}}E_3E_2E_1A = A}$$

Lower triangular.

Thus, we observe the following fact in the Gaussian elimination process:

Summary: For any $\underbrace{\textbf{regular}}_{\textit{no zero pivot}}$ matrix A, we can multiply it on the left by a sequence of elementary matrices $E_1, \ldots, E_m$, so that the product is an upper triangular matrix U, namely:

$$E_m E_{m-1} \cdots E_1 A = U$$

Then $$\underbrace{(E_1^{-1} \cdots E_m^{-1})\, E_m E_{m-1} \cdots E_1}_{I_n} A = (E_1^{-1} \cdots E_m^{-1})\, U.$$

$$A = \underbrace{(E_1^{-1} \cdots E_m^{-1})}_{L\text{, lower triangular matrix.}}\, U.$$

Fact 1:

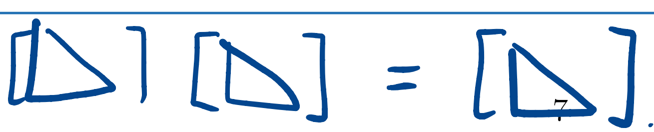
(1) We have shown that any **regular** matrix A can be factored as

$$A = LU, \qquad \text{where } U \text{ is upper triangular and } L \text{ is lower triangular.}$$

Furthermore, L has 1's on its main diagonal, and U has non-zero elements on its main diagonal (the pivots of A).

(2) $L, \tilde{L}$ are $n \times n$ lower triangular matrices, so is $L\tilde{L}$.

(3) $U, \tilde{U}$ are $n \times n$ upper triangular matrices, so is $U\tilde{U}$.



Example 4: Find LU factorization of the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -5 & 3 \\ 1 & 4 & -2 \end{pmatrix}, \quad L = I_3.$$

$$A \xrightarrow[E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{(2) - 4(1)} \overset{E_1 A}{\overset{\parallel}{\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 1 & 4 & -2 \end{pmatrix}}}, \quad \underline{L = E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\vdots$$

$$\longrightarrow \begin{pmatrix} \ \ \end{pmatrix} = U$$

To be continued!

Poll Question: Which of the following is "linear" system?

$A)$

$$\begin{cases} x + 2y + 2z &= 2 \\ 10y - z &= 1 \\ 4x + 4z^2 &= 0 \end{cases}$$

$B)$

$$\begin{cases} x + 2^4 y + 3z &= 1 \\ 2x + 10y &= -2 \\ 4x + 11^3 y &= 1 \end{cases}$$

Caution: After clicking submit, you will NOT be able to resubmit your answer!

* You should be able to see the pop up Zoom question. Answer the question by clicking the corresponding answer and then submit.