Lecture 20: Quick review from previous lecture

- An $n \times n$ matrix K is called **positive definite** if
  - it is symmetric, $K^T = K$
  - satisfies the positivity condition

$$\mathbf{x}^T K \mathbf{x} > 0 \qquad \text{for all } \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n.$$

  We write $\boxed{K > 0}$ to mean that K is **positive definite** matrix.

- Identify any $n \times n$ positive definite matrix:
  An n-by-n matrix A is **positive definite** if and only if it is:
  (a) symmetric;
  (b) regular, hence $A = LDL^T$; and
  (c) D has all positive diagonal entries, i.e. A has positive pivots.

  $A \longrightarrow \begin{bmatrix} u_{11} & & \\ & \ddots & \\ & & u_{nn} \end{bmatrix}, \quad u_{ii} > 0$

- In particular, we have another way to identify 2×2 positive definite matrix:
  $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is **positive definite** if and only if

$$a > 0 \quad \text{and} \quad \underbrace{ac - b^2}_{= \det A} > 0$$

Today we will discuss

- Sec. 3.4 - 3.5 Positive definite matrix.

- Lecture will be recorded -

Definition: If a matrix A satisfies $\mathbf{x}^T A\mathbf{x} \geq 0$ for all vectors $\mathbf{x}$, it is called **positive semidefinite**. Denote $\underline{A \geq 0}$ meaning A is positive semidefinite.

Remark: Every positive definite matrix is also positive semidefinite; but the converse is not true:

$$\boxed{\text{positive definite} \Rightarrow \text{positive semidefinite}}$$

$\Leftarrow\!\!\!\!\not\;$

$x^TAx > 0 \ \forall x \neq 0$ $\qquad$ $x^TAx \geq 0 \ \forall x$

since a positive semidefinite matrix A might have $\mathbf{x}^T A\mathbf{x} = 0$ for $\mathbf{x} \neq \mathbf{0}$.

Example 5. The matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite, but not positive definite.

(1) $A = A^T$

(2) quadratic form

$$q(x) = x^TAx = (x_1\ x_2)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - 2x_1x_2 + x_2^2$$
$$= (x_1 - x_2)^2$$
$$\geq 0$$

when $x_1 = x_2$, $q(x) = 0$.

Then $A \geq 0$ (posi. semidefinite), but not posi. definite.

Definitions:

- a matrix A is **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- Similarly, a matrix A is **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x}$.
- If a matrix is neither positive or negative semidefinite, it is called **indefinite**. This means that there are vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{y}^T A\mathbf{y} < 0$.

*Only "positive definite" matrices define inner products, via $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T A\mathbf{y}$.

§ Constructing positive definite or positive semidefinite matrices

Definition: Let V be an inner product space. The **Gram matrix** for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the matrix K given by

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix}_{n\times n}$$

Clearly K is symmetric. since symmetry property of the inner product, $\langle v_i, v_j \rangle = \langle v_j, v_i \rangle$.

Fact 5: (1) All Gram matrices are **positive semidefinite**;
(2) Gram matrices are **positive definite** precisely when the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Dot product $\langle x, y \rangle = x^T y = y^T x$.

Example 6. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} v_1^T v_1 & v_1^T v_2 \\ v_2^T v_1 & v_2^T v_2 \end{bmatrix} = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} [v_1 \ v_2] = A^T A \quad \text{where } A = [v_1 \ v_2]$$

(1) For the usual (Dot product) inner product, the Gram matrix for vectors $\mathbf{v}_1, \mathbf{v}_2$ is:

$$K = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

By Fact 5(2), since $\{v_1, v_2\}$ are l. indep, $K > 0$.

Or $6 > 0$, $\det K = 6 \cdot 1 - 2^2 = 2 > 0$. Then $K > 0$.

(2) Find the Gram matrix for vectors $\mathbf{v}_1, \mathbf{v}_2$ with respect to $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T D \mathbf{y}$, where $D = \text{diag}(3, 2, 1)$.

$$K = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{bmatrix} = \begin{bmatrix} v_1^T D v_1 & v_1^T D v_2 \\ v_2^T D v_1 & v_2^T D v_2 \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ 4 & 2 \end{bmatrix}$$

$$v_1^T D v_1 = (1\ 2\ 1) \begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (1\ 2\ 1) \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 3 + 8 + 1 = 12$$

$\ulcorner 12>0$, $\det K>0$. So $K>0$ $\lrcorner$.

Then $K = A^T D A$, where $A = [v_1 \ v_2]$.

*We write $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. The entry (i,j) of A^TA is $\mathbf{v}_i^T\mathbf{v}_j = \langle \mathbf{v}_i, \mathbf{v}_j\rangle$. Then A^TA is Gram matrix. By Fact 5, we have

Fact 6: Suppose A is any $m\times n$ matrix. Then $K = A^TA$ is positive semidefinite.

We can also prove it directly.

① $K^T = (A^TA)^T = A^T(A^T)^T = A^TA = K$.
K is symmetric.

② $q(x) = x^TKx = x^TA^TAx = (Ax)^TAx$
$= \|Ax\|_2^2 \quad (\|y\|_2^2 = \sum y_i^2)$
$\geq 0, \ \forall x$

Then $K \geq 0$.

Observation: To have $K>0$ we need $x^TKx = 0 \iff x = 0$.

$x^TKx = 0 \iff Ax = 0$.
$\iff x \in \ker A$.

If $\ker A = \{0\}$, then $x = 0$. #.

Thus we have

Fact 7: $K = A^TA$ is **positive definite** if and only if
1) the rank of $A_{m\times n}$ is n (in particular, we must have $n \leq m$);
2) the columns of A are linearly independent;
3) $\ker A = \{\mathbf{0}\}$.

$\mathbb{R}^n \xrightarrow{A_{m\times n}} \mathbb{R}^m$

$\ker A = \{0\}$. $\qquad$ $\operatorname{rank}(A) = n$.

Recall that from **Fact 1:** Every inner product on $\mathbb{R}^m$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T C \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^m, \tag{1}$$

where C is a **positive definite** $m \times m$ matrix.

Then the Gram matrix of $\mathbf{v}_1, \cdots, \mathbf{v}_n$ with respect this inner product (1) is

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \cdots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix} = \begin{bmatrix} v_1^T C v_1 & \cdots & v_1^T C v_n \\ & & \vdots \\ v_n^T C v_1 & & v_n^T C v_n \end{bmatrix}$$

Therefore, in this case, if $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is any m-by-n matrix, then

$$K = A^T C A.$$

$= A^T C A$, where $A = [v_1 \cdots v_n]$

Similar to **Fact 7**, we have

Fact 8: Let C be a positive definite matrix.
(1) $K = A^T C A$ is always positive semidefinite,
(2) $K = A^T C A$ is positive definite if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent (i.e. $\ker A = \{\mathbf{0}\}$).

(1) It's clear that $K^T = K$.

$q(x) = x^T K x = x^T A^T C A x = (Ax)^T C (Ax)$

> 0 if $z = Ax \neq 0$

$= 0$ if $z = Ax = 0$

So, $q(x) \geq 0$ $\forall x$. implies $K \geq 0$ since $C > 0$. #.

(2) If $\ker A = \{0\}$, then $Ax = 0$ gives $x = 0$.

Thus $x^T K x = 0 \iff x = 0$. #.

Fact 9: Let $K = A^T C A$, where A is an $m \times n$ matrix and C is an $m \times m$ positive definite matrix. Then

$$\ker K = \ker A$$

and moreover $\mathsf{rank}(K) = \mathsf{rank} A$.

To be continued!

Example 7. Consider the vector space $C^0([0,1])$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Construct the Gram matrix K corresponding to $1, x, x^2$. Is K positive definite?

$$K = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \langle x, x^2 \rangle \\ \langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle \end{bmatrix}_{3\times 3} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

Compute

$$\langle x^i, x^j \rangle = \int_0^1 x^i \cdot x^j dx = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1} x^{i+j+1} \Big|_0^1 = \frac{1}{i+j+1}$$

By Fact 5 (2), since $\{1, x, x^2\}$ are l. indep, we have $\underline{K > 0}$. #

§ To find the symmetric matrix A from the quadratic form

The quadratic form $\mathbf{x}^T A\mathbf{x}$ defined by the symmetric matrix $A = (a_{ij})$, $a_{ij} = a_{ji}$ (square, of size n-by-n) is

$$\boxed{\mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j}$$

Q: How do we go "backwards" to find the symmetric A from the quadratic form $\mathbf{x}^T A\mathbf{x}$?

Example 8. Determine if the following quadratic form is positive definite (that is, $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$).

1. In 2 dimensions, suppose

$$\mathbf{x}^T A\mathbf{x} = 2x_1^2 - 6x_1x_2 + 3x_2^2.$$

Then $A = \begin{bmatrix} 2 & -3 \\ -3 & 3 \end{bmatrix}$ divide by 2

$2 > 0$, $\det A = 2\cdot 3 - (-3)^2 < 0$.

A is not posi. definite. quadratic form is not positive definite.

2. In 3 dimensions, suppose

$$\mathbf{x}^T K\mathbf{x} = x_1^2 + 4x_1x_2 - 2x_1x_3 + 6x_2^2 + 7x_3^2.$$

Then

$$K = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then $K > 0$. so is its quadratic form.

Poll Question 1: Is the matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}$ positive definite?

A) Yes
B) No

Poll Question 2: Is the matrix $B = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ positive definite?

A) Yes
B) No

Recall: to identify 2×2 positive definite matrix:
$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is **positive definite** if and only if

$$a > 0 \quad \text{and} \quad ac - b^2 > 0$$

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