Lecture 21: Quick review from previous lecture

- Let V be any inner product space. The **Gram matrix** for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is given by

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_n \rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1 \rangle & \langle \mathbf{v}_2, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_2, \mathbf{v}_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{v}_n, \mathbf{v}_1 \rangle & \langle \mathbf{v}_n, \mathbf{v}_2 \rangle & \ldots & \langle \mathbf{v}_n, \mathbf{v}_n \rangle \end{pmatrix}$$

  - Gram matrices are always **positive semidefinite**
  - they are **positive definite** precisely when the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Today we will discuss

- Continue Sec. 4.1 Orthogonal(Orthonormal) bases.

- Lecture will be recorded -

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Gram matrix with $\langle x, y\rangle = x^T C y$, C posi defi.

Fact 9: Let $K = A^T C A$, where A is an $m \times n$ matrix and C is an $m \times m$ positive definite matrix. Then

$$(1) \quad \ker K = \ker A$$

and, moreover, (2) $\mathsf{rank}(K) = \mathsf{rank} A$.

(1) Show $\ker A = \ker K$.

① $\ker A \subset \ker K$

$x \in \ker A$, $Ax = 0$. Then $Kx = A^T C A x = 0$. yields $x \in \ker K$.

② $\ker K \subset \ker A$

$x \in \ker K$. $Kx = 0$ implies $A^T C A x = 0$.

Then $x^T A^T C A x = 0$.

$\quad\quad (Ax)^T C (Ax)$

Since $C > 0$ (posi. defi.), one gets $Ax = 0$.

So $x \in \ker A$.

By ① ②, we get $\ker K = \ker A$.

(2) We have learn that $A_{m \times n}$.

$\mathrm{rank}(A) + \dim(\ker A) = n$.

$\mathrm{rank}(K) + \dim(\ker K) = n$.

By (1), $\dim(\ker A) = \dim(\ker K)$ and

thus, $\mathrm{rank}(A) = \mathrm{rank}(K)$. #

4 Orthogonality

4.1 Orthogonal and Orthonormal Bases

We've already seen that in an inner product space V, two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal** if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Definition: Let V be an inner product space. We say that nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **(mutually) orthogonal** if any two vectors are orthogonal, that is,

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \quad \text{if } i \neq j$$

If additionally $\|\mathbf{v}_i\| = 1$, we say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **orthonormal**.

Definition: Let V be an inner product space with $\dim V = n$.

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are 1) mutually orthogonal vectors; 2) also a basis[a] for V, then we say they are an **orthogonal basis**.
- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are 1) orthonormal; 2) a basis for V, then we say they are an **orthonormal basis**.

[a] A basis for a vector space V is a ① linearly independent set of vectors that ② span V.

Example 1. (1) In $\mathbb{R}^n$ equipped with the standard dot product, an *orthonormal basis* is the standard basis:

$$\mathbf{e}_1 = (1, 0, \ldots, 0, 0)^T, \qquad \mathbf{e}_2 = (0, 1, \ldots, 0, 0)^T, \qquad \mathbf{e}_n = (0, 0, \ldots, 0, 1)^T$$

$\{e_1, \ldots, e_n\}$ is a basis.

$\langle e_i, e_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ $\qquad \|e_i\| = \sqrt{\langle e_i, e_i \rangle} = 1, \ 1 \le i \le n.$

(2) The set

$$\{(1, 2)^T, (2, -1)^T\}$$

$v_1 \qquad v_2$

is an orthogonal basis for $\mathbb{R}^2$ with the standard dot product. Use this basis to get an orthonormal basis for $\mathbb{R}^2$.

$\langle \binom{1}{2}, \binom{2}{-1} \rangle = 0.$

An orthonormal basis $\left\{ \frac{1}{\sqrt{5}} \binom{1}{2}, \frac{1}{\sqrt{5}} \binom{2}{-1} \right\}.$

$\|v_1\| = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{\langle \binom{1}{2}, \binom{1}{2} \rangle} = \sqrt{5}.$

Fact 1: Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthogonal basis. Then

$$\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$$

form an orthonormal basis.

$\to \langle v_i, v_j \rangle = 0, \ i \neq j.$

Fact 2: If nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **mutually orthogonal**, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

[To see this] Consider $a_1 v_1 + \cdots + a_n v_n = 0$. Show $a_1 = \cdots = a_n = 0$.

Inner product with v_1:

$$0 = \langle 0, v_1 \rangle = \langle a_1 v_1 + \cdots + a_n v_n, v_1 \rangle$$

(since $\langle v_i, v_j \rangle = 0$, $i \neq j$)

$$= a_1 \langle v_1, v_1 \rangle + \overset{0}{\cdots} + \overset{0}{a_n \langle v_n, v_1 \rangle}.$$

$$= a_1 \|v_1\|^2$$

Since $\|v_1\| \neq 0$, one gets $a_1 = 0$.

Similarly, inner product with v_i, we get $a_i = 0$, $2 \le i \le n$.

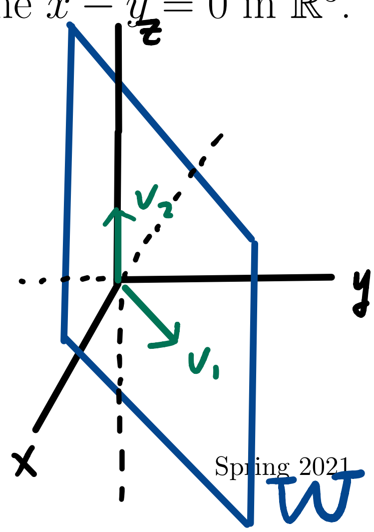
Fact 2 directly implies the following result.

Fact 3: Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero, mutually orthogonal (resp. orthonormal) vectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal (resp. orthonormal) basis for their span $W = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Example 2. $\mathbf{v}_1 = (1, 1, 0)^T$ and $\mathbf{v}_2 = (0, 0, 1)^T$ are vectors in $\mathbb{R}^3$ under the standard dot product. The space $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane $x - y = 0$ in $\mathbb{R}^3$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W.

$$\langle v_1, v_2 \rangle = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle = 0.$$

So v_1, v_2 are orthogonal.



$\xrightarrow{\text{Fact 3}}$ $W = \text{span}\{v_1, v_2\}$ has an orthogonal basis $\{v_1, v_2\}$.

Example 3. Explain why vectors $\mathbf{x} = (1, 1, 0)^T$, $\mathbf{y} = (1, -1, 1)^T$, and $\mathbf{z} = (1, -1, -2)^T$ form an orthogonal basis in $\mathbb{R}^3$ under the standard dot product? Turn them into an orthonormal basis.

$\|x\|^2 = \langle x, x \rangle = 2$

$\|x\| = \sqrt{2}$

(1) $\langle x, y \rangle = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle = 0$

$\langle x, z \rangle = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix} \right\rangle = 0$

$\langle y, z \rangle = \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix} \right\rangle = 1 + 1 - 2 = 0.$

So $\{x, y, z\}$ are orthogonal. By Fact 2, $\{x, y, z\}$ are l. indep. Combining $\dim(\mathbb{R}^3) = 3$, $\{x, y, z\}$ is an orthogonal basis #.

(2) To get orthonormal basis.

$\left\{ \frac{x}{\|x\|}, \frac{y}{\|y\|}, \frac{z}{\|z\|} \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix} \right\}$ #

§ Computations in Orthogonal Bases

Q: What are the advantages of orthogonal (orthonormal) bases?

It is simple to find the *coordinates of a vector* in the orthogonal (orthonormal) basis. However, in general this is not so easy if it is not in such basis.

Fact 4: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an **orthogonal basis** in any inner product space V, then for any vector $\mathbf{v} \in V$, we have

$$\mathbf{v} = \underbrace{\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}}_{a_1} \mathbf{v}_1 + \ldots + \underbrace{\frac{\langle \mathbf{v}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}}_{a_n} \mathbf{v}_n.$$

Moreover, we have

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} \left(\frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|} \right)^2.$$

Let $a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for $i = 1, \ldots, n$. We call $(a_1, \ldots, a_n)^T$ the coordinates of $\mathbf{v}$ in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

[To see this]

continue next time

Fact 5: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an **orthonormal basis** in any inner product space V, then for any vector $\mathbf{v} \in V$, we have $\|v_i\| = 1$.

$$\|v_i\|=1 \quad \text{Fact 4} \Rightarrow \quad \mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \ldots + \langle \mathbf{v}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Moreover, we have

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} \left(\langle \mathbf{v}, \mathbf{v}_i \rangle \right)^2.$$

Example 4. Consider the same orthogonal basis as in Example 3:

$$\mathbf{x} = (1,\, 1,\, 0)^T, \quad \mathbf{y} = (1,\, -1,\, 1)^T, \quad \mathbf{z} = (1,\, -1,\, -2)^T.$$

Write $\mathbf{v} = (1,\, 0,\, 3)^T$ as the linear combination of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^3$ under the standard dot product. (that is, find the coordinates of $\mathbf{v}$ in this orthogonal basis.)

$$v = a_1 x + a_2 y + a_3 z.$$

$$a_1 = \frac{\langle v, x \rangle}{\|x\|^2} = \frac{\left\langle \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle} = \frac{1}{2}$$

$$a_2 = \frac{\langle v, y \rangle}{\|y\|^2} = \frac{\langle \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \rangle}{\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \rangle} = \frac{4}{3}.$$

$$a_3 = \frac{\langle v, z \rangle}{\|z\|^2} = \frac{-5}{6}.$$

Then $v = \frac{1}{2} x + \frac{4}{3} y - \frac{5}{6} z$. #

Example 5. We consider the inner product space $\mathcal{P}^{(2)}([0,1])$ (the set of polynomials of degree ≤ 2) equipped with the L^2 inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ in the following problems.

(1) The basis $1, x, x^2$ do NOT form an orthogonal basis in $\mathcal{P}^{(2)}([0,1])$.

$$\langle 1, x \rangle = \int_0^1 1 \cdot x \, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2} \neq 0.$$

(2) $\{p_1(x) = 1,\ p_2(x) = x - \frac{1}{2},\ p_3(x) = x^2 - x + \frac{1}{6}\}$ is an orthogonal basis of $\mathcal{P}^{(2)}([0,1])$.

$$\langle p_1, p_2 \rangle = \int_0^1 1 \left(x - \tfrac{1}{2}\right) dx = \tfrac{1}{2} x^2 - \tfrac{1}{2} x \Big|_0^1 = 0.$$

$$\langle p_1, p_3 \rangle = \int_0^1 1 \left(x^2 - x + \tfrac{1}{6}\right) dx = \tfrac{1}{3} x^3 - \tfrac{1}{2} x^2 + \tfrac{1}{6} x \Big|_0^1 = 0.$$

$$\langle p_2, p_3 \rangle = \sim = 0.$$

Since $\{p_1, p_2, p_3\}$ are orthogonal, Fact 2 implies $\{p_1, p_2, p_3\}$ are l. indep. Together with $\dim(\mathcal{P}^{(2)}([0,1])) = 3$, $\{p_1, p_2, p_3\}$ is an orthogonal basis for $\mathcal{P}^{(2)}([0,1])$. #

Poll Question 1: Are $\{(1,0,1)^T, (1,0,-1)^T\}$ orthogonal with respect to the standard dot product

A) Yes
B) No

Poll Question 2: Are $\{(1,0,1)^T, (1,0,-1)^T\}$ orthogonal with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \cancel{\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y}}$

$$x^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} y$$

A) Yes
B) No

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