Lecture 22: Quick review from previous lecture

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero vectors that are **mutually orthogonal**, meaning $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$,

  $< v_i, v_j > = 0 \ , i \neq j$

  $$\text{mutually orthogonal} \Rightarrow \ \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly independent}$$

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for V and orthogonal (orthonormal), we call they are **orthogonal (orthonormal) basis**.

- If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an "orthogonal" basis in any inner product space V, then for any vector $\mathbf{v} \in V$ we have

  $$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n,$$

  where the coordinates of $\mathbf{v}$ ~~is~~ in this basis is given by

  $(a_1, \ldots, a_n)^T$

  $$a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}, \ i = 1, \cdots, n.$$

Today we will discuss

- Sec. 4.1 Orthogonal(Orthonormal) bases.
- Sec. 4.2 The Gram-Schmidt process.

- Lecture will be recorded -

- HW7 due today at 6pm.

It is simple to find the *coordinates of a vector* in the orthogonal (orthonormal) basis. However, in general this is not so easy if it is not in such basis.

Fact 4: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an **orthogonal basis** in any inner product space V, then for any vector $\mathbf{v} \in V$, we have

$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \ldots + \frac{\langle \mathbf{v}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Moreover, we have

$$\|\mathbf{v}\|^2 = \sum_{i=1}^{n} \left(\frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|} \right)^2.$$

Let $a_i = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$ for $i = 1, \ldots, n$. We call $(a_1, \ldots, a_n)^T$ the coordinates of $\mathbf{v}$ in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

[To see this] (1) Since $\{v_1, \ldots, v_n\}$ is a basis, for any vector $v \in V$, we can express v as

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Inner product with v_1:

$$\langle v, v_1 \rangle = \langle a_1 v_1 + \cdots + a_n v_n, v_1 \rangle$$
$$= a_1 \langle v_1, v_1 \rangle + a_2 \underbrace{\langle v_2, v_1 \rangle}_{0} + \cdots + a_n \underbrace{\langle v_n, v_1 \rangle}_{0}$$
$$= a_1 \|v_1\|^2$$

since $\langle v_i, v_j \rangle = 0$, $i \neq j$

Then $a_1 = \dfrac{\langle v, v_1 \rangle}{\|v_1\|^2}$. Similarly, $a_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$, $i = 2, \ldots, n$.

(2) $$\|v\|^2 = \langle v, v \rangle = \langle a_1 v_1 + \cdots + a_n v_n, a_1 v_1 + \cdots + a_n v_n \rangle$$
$$= a_1^2 \langle v_1, v_1 \rangle + \cdots + a_n^2 \langle v_n, v_n \rangle$$
$$= \left(\frac{\langle v, v_1 \rangle}{\|v_1\|^2} \right)^2 \|v_1\|^2 + \cdots + \left(\frac{\langle v, v_n \rangle}{\|v_n\|^2} \right)^2 \|v_n\|^2$$
$$= \frac{\langle v, v_1 \rangle^2}{\|v_1\|^2} + \cdots + \frac{\langle v, v_n \rangle^2}{\|v_n\|^2} \#$$

Example 5. We consider the inner product space $\mathcal{P}^{(2)}([0,1])$ (the set of polynomials of degree ≤ 2) equipped with the L^2 inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$ in the following problems.

(1) The basis $1, x, x^2$ do NOT form an orthogonal basis in $\mathcal{P}^{(2)}([0,1])$.

(2) $\{p_1(x) = 1, p_2(x) = x - \frac{1}{2}, p_3(x) = x^2 - x + \frac{1}{6}\}$ is an orthogonal basis of $\mathcal{P}^{(2)}([0,1])$.

[(1) and (2) were discussed in Lecture 21]

(3) Write $p(x) = x^2 + x + 1$ in terms of the basis p_1, p_2, p_3 in (2).

$$p(x) = \underbrace{\frac{\langle p, p_1\rangle}{\|p_1\|^2}}_{11/6} p_1 + \underbrace{\frac{\langle p, p_2\rangle}{\|p_2\|^2}}_{2} p_2 + \underbrace{\frac{\langle p, p_3\rangle}{\|p_3\|^2}}_{1} p_3 .$$

① $\langle p, p_1\rangle = \int_0^1 (x^2+x+1)\, 1\, dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \Big|_0^1 = 11/6 .$

$\|p_1\|^2 = \langle p_1, p_1\rangle = \int_0^1 1^2\, dx = 1 .$

② $\langle p, p_2\rangle = \int_0^1 (x^2+x+1)(x-\frac{1}{2})\, dx = 1/6$

$\|p_2\|^2 = \langle p_2, p_2\rangle = \int_0^1 (x-\frac{1}{2})^2\, dx = 1/12$

$a_2 = \frac{\langle p, p_2\rangle}{\|p_2\|^2} = \underline{2}$ #

③ $a_3 = \frac{\langle p, p_3\rangle}{\|p_3\|^2} = 1 .$ Then $\underline{p = \frac{11}{6}p_1 + 2p_2 + p_3}$ #

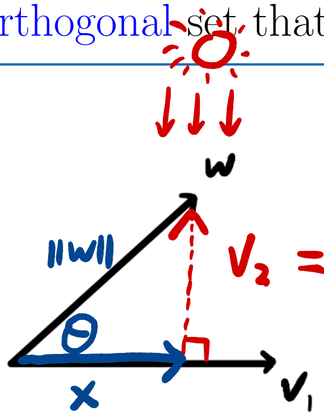
(See also Ex 4.11 in P. 190 in Textbook)

4.2 The Gram-Schmidt Process

Q: How can we construct the orthogonal (or orthonormal) bases?
This can be done by the algorithm, known as **the Gram-Schmidt process**.

§ Given 1 nonzero vectors $\mathbf{v}_1$ and another vector $\mathbf{w}$

Q: How do we make up an vector $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ so that it forms an orthogonal set that spans the same subspace as span$\{\mathbf{v}_1, \mathbf{w}\}$?



$$v_2 = w - x = w - \frac{\langle w, v_1\rangle v_1}{\|v_1\|^2}$$

Then $\langle v_2, v_1\rangle = 0$.
and $\text{span}\{v_1, v_2\} = \text{span}\{v_1, w\}$.

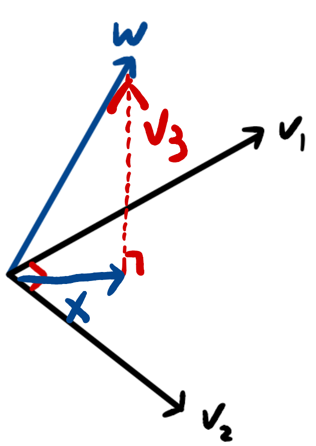
$$\|x\| = \|w\| \cos\theta = \cancel{\|w\|} \frac{\langle w, v_1\rangle}{\cancel{\|w\|}\,\|v_1\|} = \frac{\langle w, v_1\rangle}{\|v_1\|}$$

$$x = \frac{\langle w, v_1\rangle}{\|v_1\|} \underbrace{\frac{v_1}{\|v_1\|}}_{\text{direction}} = \frac{\langle w, v_1\rangle}{\|v_1\|^2} v_1$$

(orthogonal projection of w onto span$\{v_1\}$)

§ Given 2 orthogonal nonzero vectors $\mathbf{v}_1, \mathbf{v}_2$, another vector $\mathbf{w}$

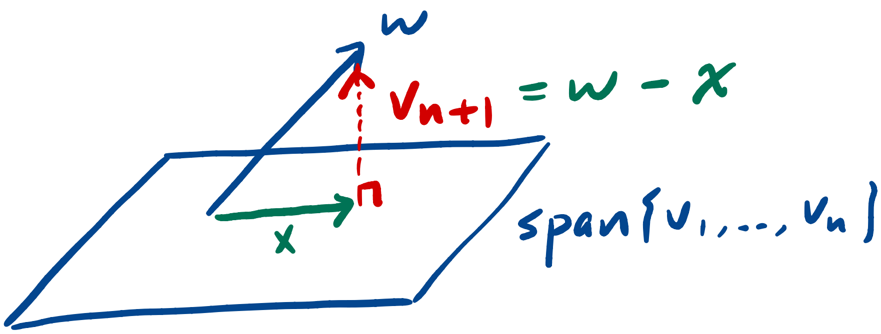
Q: How do we make up an vector $\mathbf{v}_3$ orthogonal to both $\mathbf{v}_1, \mathbf{v}_2$? In particular, the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans the same subspace as span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$?



$$v_3 = w - x = w - \left(\frac{\langle w, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w, v_2\rangle}{\|v_2\|^2} v_2\right)$$

$$x = \frac{\langle w, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle w, v_2\rangle}{\|v_2\|^2} v_2$$

(orthogonal projection of w onto span$\{v_1, v_2\}$)



§ General case

Given nonzero orthogonal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then for any vector $\mathbf{w}$,

$$\mathbf{x} = \boxed{\sum_{i=1}^{n} \frac{\langle \mathbf{w}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \mathbf{v}_i}$$

is called the **orthogonal projection** of $\mathbf{w}$ onto $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
* Note that $\mathbf{x}$ is the vector nearest to $\mathbf{w}$ in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Also we have

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}\}.$$

Fact 1: The vector

$$\mathbf{v}_{n+1} = \mathbf{w} - \mathbf{x} = \mathbf{w} - \left(\sum_{i=1}^{n} \frac{\langle \mathbf{w}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \mathbf{v}_i \right)$$

is **orthogonal** to each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Check: $\langle v_{n+1}, v_1 \rangle = 0, \ldots, \langle v_{n+1}, v_n \rangle = 0$ #

§ The Gram-Schmidt process

We start with any basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ for the inner product space V.

We then orthogonalize each one to the preceding ones, building up an "orthogonal basis" as we go.

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{w}_1 \\
\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\langle \mathbf{w}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{w}_3 - \frac{\langle \mathbf{w}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{w}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&\vdots \qquad \vdots \\
\mathbf{v}_n &= \mathbf{w}_n - \frac{\langle \mathbf{w}_n, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \ldots - \frac{\langle \mathbf{w}_n, \mathbf{v}_{n-1} \rangle}{\|\mathbf{v}_{n-1}\|^2}\mathbf{v}_{n-1}
\end{aligned}$$

orthogonal projection of w_2 onto span$\{v_1\}$.

Then

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \quad \text{is an orthogonal basis}$$

and, moreover,

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \ldots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}\right\} \quad \text{is an orthonormal basis}$$

orthonormal basis $\{u_i\}$:

① $\langle u_i, u_j \rangle = 0$, $i \neq j$

② $\|u_i\| = 1$

Example 1. Consider the vectors

$$\mathbf{w}_1 = (1,\ 1,\ 0)^T, \quad \mathbf{w}_2 = (0,\ 1,\ 1)^T, \quad \mathbf{w}_3 = (1,\ 0,\ 1)^T$$

that form a basis of $\mathbb{R}^3$ under the standard dot product. To construct an orthogonal basis and an orthonormal basis using the Gram-Schmidt process.

By Gram Schmidt, we get

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$V_3 = W_3 - \frac{\langle W_3, V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle W_3, V_2 \rangle}{\|V_2\|^2} V_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \underbrace{\frac{1/2}{3/2}}_{=1/3} \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 - \frac{1}{2} + \frac{1}{6} \\ 0 - \frac{1}{2} - \frac{1}{6} \\ 1 - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

Then $\{V_1, V_2, V_3\}$ is an orthogonal basis

Turn them into orthonormal basis:

$$\left\{ \frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} \right\}.$$

$$= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \sqrt{\frac{2}{3}} \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}, \frac{\sqrt{3}}{2} \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix} \right\} \#$$