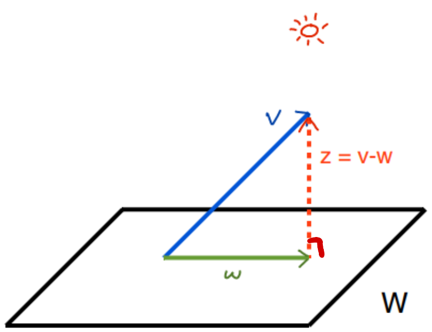
Lecture 25: Quick review from previous lecture

- $$\underbrace{\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}}_{A(nonsingular)} = \underbrace{\begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{pmatrix}}_{Q \text{ (orthogonal)}} \underbrace{\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{pmatrix}}_{R}$$

  where $r_{kk} = \|\mathbf{v}_k\| = \langle \mathbf{a}_k, \mathbf{q}_k \rangle$ and $r_{ij} = \langle \mathbf{a}_j, \mathbf{q}_i \rangle$. This is called the **QR factorization**.

- The **orthogonal projection** of $\mathbf{v}$ onto the subspace W of V is the element $\mathbf{w} \in W$ such that the difference $\mathbf{z} = \mathbf{v} - \mathbf{w}$ orthogonal to W.
  Moreover, let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is an orthogonal basis of W. Then

$$\mathbf{w} = \frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \cdots + \frac{\langle \mathbf{v}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n.$$



w : orthogonal projection of v onto space W

z is orthogonal to W.

Today we will discuss

- Sec. 4.4 Orthogonal Projections

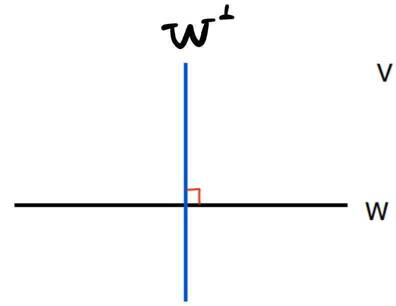
- Lecture will be recorded -

- HW8 due today at 6pm.

Definition: If W is a subspace of an inner product space V, its **orthogonal complement** $W^{\perp}$ (pronounced "W perp") is the set of all vectors orthogonal to W, that is,

$$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

- $W^{\perp}$ is also a subspace of V.
- If $W = \text{span}\{\mathbf{w}\}$, we will also denote $W^{\perp}$ by $\mathbf{w}^{\perp}$.
- Note that the "only vector" contained in both W and $W^{\perp}$ is $\mathbf{0}$.



$W \cap W^{\perp} = \{0\}$.

Example 2. Let $\mathbf{w}_1 = (1, 2, 1)^T$ and $\mathbf{w}_2 = (0, -1, 1)^T$.

(1) Suppose $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ in $\mathbb{R}^3$ under the usual dot product. Find $W^{\perp}$.

For $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $W^{\perp}$,

$\langle w_1, \vec{x} \rangle = 0$

$\langle w_2, \vec{x} \rangle = 0$

$\Rightarrow \begin{cases} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \\ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \end{cases} \Rightarrow \begin{cases} a + 2b + c = 0 \\ -b + c = 0 \end{cases}$

homogeneous l. system.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b = c \\ a = -2b - c = -3c. \end{cases}$$

$W^{\perp} = \left\{ \begin{pmatrix} -3c \\ c \\ c \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}$

(2) Suppose $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ in $\mathbb{R}^3$ under the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T D \mathbf{y}$, where $D = \text{diag}(1, 2, 2)$. Find $W^{\perp}$.

For $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W^{\perp}$.

$\langle w_1, \vec{x} \rangle = 0$

$\langle w_2, \vec{x} \rangle = 0$

$\Rightarrow \begin{cases} (1\ 2\ 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \\ (0\ -1\ 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \end{cases} \Rightarrow \begin{cases} a + 4b + 2c = 0 \\ -2b + 2c = 0. \end{cases}$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} b = c \\ a = -4b - 2c = -6c. \end{cases}$$

$W^{\perp} = \left\{ \begin{pmatrix} -6c \\ c \\ c \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}.$

Fact 4: If W is a subspace of an inner product space V with $\dim W = n$ and $\dim V = m$, then every vector $\mathbf{v} \in V$ can be **uniquely** decomposed into

$$\mathbf{v} = \mathbf{w} + \mathbf{z}, \qquad \text{where } \mathbf{w} \in W \text{ and } \mathbf{z} \in W^{\perp}.$$

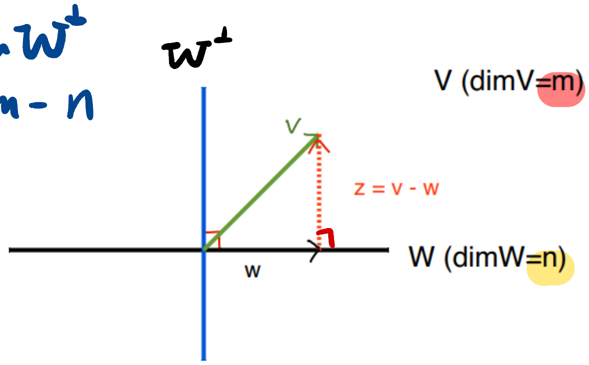
Moreover, we have

$$\dim W^{\perp} = m - n$$

and thus,

$$\dim V = \dim W + \dim W^{\perp}.$$

$\dim W^{\perp} = m - n$



Example 3. Let $W = \text{img}\, A$, where $A = \begin{array}{c} \begin{matrix} v_1 & v_2 & \; \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \end{array}$.

(1) Find $W^{\perp}$, that is, $(\text{img}\, A)^{\perp}$ with respect to the dot product.

① Find $W = \text{img}\, A$:

$$A \xrightarrow{③-①} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{③+②} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then a basis for img A is $\{v_1, v_2\}$.

$W = \text{span}\{v_1, v_2\}$.

② Find $W^{\perp}$: $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W^{\perp}$.

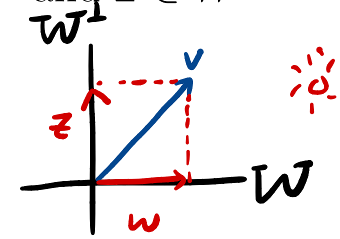
$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \Rightarrow \begin{matrix} a = -c \\ b = c \end{matrix}$$

$W^{\perp} = \left\{ \begin{pmatrix} -c \\ c \\ c \end{pmatrix} \middle| c \in \mathbb{R} \right\}$ $= \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

NOTE: $\dim W + \dim W^{\perp} = 2 + 1 = 3 \; (= \dim \mathbb{R}^3)$.

(2) For $\mathbf{v} = (1,2,3)^T$. Decompose $\mathbf{v}$ into $\mathbf{v} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$ with respect to the dot product.



Find z (orthogonal projection of v onto $W^\perp$)

$$z = \frac{\left\langle v, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rangle}{\left\langle \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rangle} \begin{pmatrix}-1\\1\\1\end{pmatrix} = \frac{4}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

$$w = v - z = \begin{pmatrix}7\\2\\5\end{pmatrix}\Big/3. \Rightarrow v = z + w. \quad \#.$$

* We don't do orthogonal projection of v onto W here since $\{v_1, v_2\}$ is NOT orthogonal yet.

> **Fact 5:** If W is a subspace of an inner product space V with $\dim W = n < \infty$, then
> $$(W^\perp)^\perp = W.$$

Example 4. Let $V = \mathcal{P}^{(4)}([-1,1])$ with L^2 inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$. Let $W = \mathcal{P}^{(1)} = \{ax + b\}$ has a basis $\{1, x\}$. $(\dim W = 2)$

(1) Find $W^\perp$. For $p = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ in $W^\perp$,

$$0 = \langle p, 1\rangle = \int_{-1}^{1} a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \, dx$$

$$= \frac{a_4}{5}x^5 + \cancel{\frac{a_3}{4}x^4} + \frac{a_2}{3}x^3 + \cancel{\frac{1}{2}a_1x^2} + a_0x \Big|_{-1}^{1}$$

$$\Rightarrow \frac{2}{5}a_4 + 2\frac{a_2}{3} + 2a_0 = 0.$$

$$0 = \langle p, x\rangle = \int_{-1}^{1} (a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)\, x \, dx$$

$$= \int_{-1}^{1} a_4x^5 + a_3x^4 + a_2x^3 + a_1x^2 + a_0x \, dx$$

$$= \cancel{\frac{a_4}{6}x^6} + \frac{1}{5}a_3x^5 + \cancel{\frac{a_2}{4}x^4} + \frac{a_1}{3}x^3 + \Big|_{-1}^{1}$$

$$\Rightarrow \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0$$

$\dim W^{\perp} = 3$

(2) Find dimension of $W^{\perp}$. and it's basis. → Exercise.

① $\begin{cases} 15a_0 + 5a_2 + 3a_4 = 0 \\ 5a_1 + 3a_3 = 0. \end{cases}$

$$\begin{bmatrix} 15 & 0 & 5 & 0 & 3 \\ 0 & 5 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Free variables: a_2, a_3, a_4.

$$W^{\perp} = \left\{ \begin{pmatrix} -\frac{1}{3}a_2 - \frac{1}{5}a_4 \\ -\frac{3}{5}a_3 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \ a_2, a_3, a_4 \in \mathbb{R} \right\}.$$

Recall that:

Suppose $A = A_{m\times n}$ is any matrix with $\mathsf{rank} A = r$. We've seen that

$$\mathsf{dim}(\text{coimg } A) = r \quad \text{and} \quad \mathsf{dim}(\ker A) = n - r.$$

Fact 6: Let A be any real $m \times n$ matrix. Then

$$\ker A = (\text{coimg } A)^{\perp} \quad (\text{and coimg } A = (\ker A)^{\perp}).$$

[To see this:]

To be continued!

Poll Question 1: Which vector is the **orthogonal projection of v** onto the space W?

A) **w**
B) **z**