Lecture 26: Quick review from previous lecture

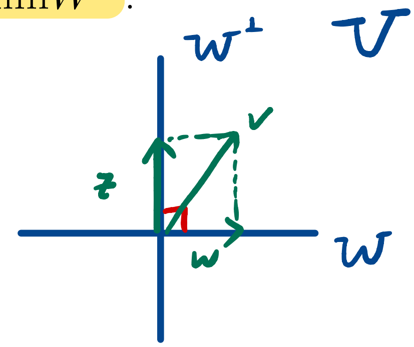
- If W is a subspace of V with $\dim W = n$ and $\dim V = m$, then every vector $\mathbf{v} \in V$ can be **uniquely** decomposed into

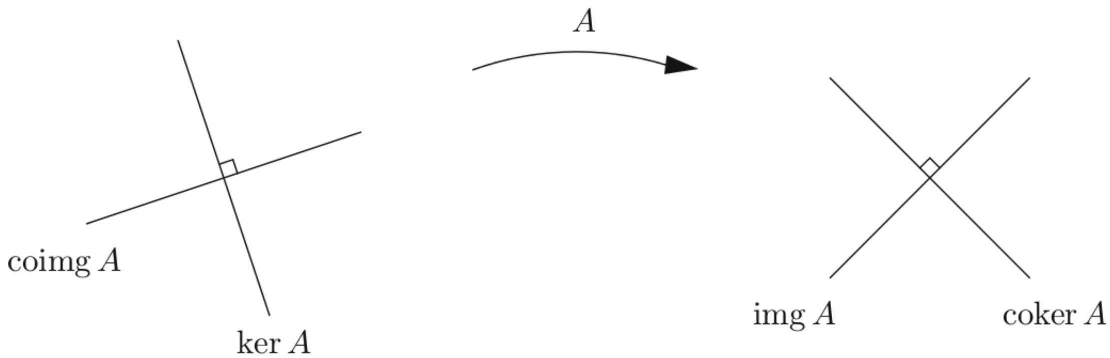
$$\mathbf{v} = \mathbf{w} + \mathbf{z}$$

m-n

where $\mathbf{w} \in W$ and $\mathbf{z} \in W^{\perp}$. Moreover, $\dim V = \dim W + \dim W^{\perp}$.

- The only vector in both W and $W^{\perp}$ is zero element $\mathbf{0}$.
- $\operatorname{coimg} A = (\ker A)^{\perp}$ and $\operatorname{img} A = (\operatorname{coker} A)^{\perp}$.





Today we will discuss

- Section 4.4 Orthogonal Projections
- Section 7.1 Linear Functions

- Lecture will be recorded -

- Exam 2 (next Wednesday 3/31) will cover 2.5, 3.1-3.5, and 4.1-4.4. Instruction and practice exam have been announced on Canvas.

Recall that: $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

$\dim(\operatorname{img} A) = \dim(\operatorname{coimg} A)$

Suppose $A = A_{m\times n}$ is any matrix with $\operatorname{rank} A = r$. We've seen that

$$\dim(\operatorname{coimg} A) = r \quad \text{and} \quad \dim(\ker A) = n - r.$$

$\operatorname{coimg} A = \operatorname{span}\{\text{rows of } A\}$

Fact 6: Let A be any real $m \times n$ matrix. Then

$$\ker A = (\operatorname{coimg} A)^{\perp} \quad (\text{and } \operatorname{coimg} A = (\ker A)^{\perp}).$$

[To see this:] $A = \begin{bmatrix} r_1^T \\ \vdots \\ r_m^T \end{bmatrix}_{m\times n}$, r_i^T : ith row of A.

$x \in \ker A$, then $\begin{bmatrix} r_1^T \\ \vdots \\ r_m^T \end{bmatrix} x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow r_k^T x = 0,\ 1 \le k \le m.$

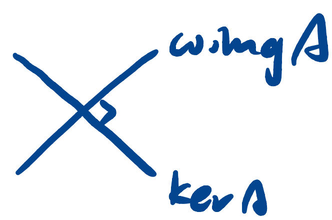
$r_k^T x = r_k \cdot x$

Then x is orthogonal to each row r_i^T.

$\Rightarrow x \perp \underline{\operatorname{span}\{r_1^T, \ldots, r_m^T\}} = \operatorname{coimg} A$

$\Rightarrow \ker A \perp \operatorname{coimg} A$

$\Rightarrow \ker A = (\operatorname{coimg} A)^{\perp}$.



Fact 5 ($(W^{\perp})^{\perp} = W$) implies

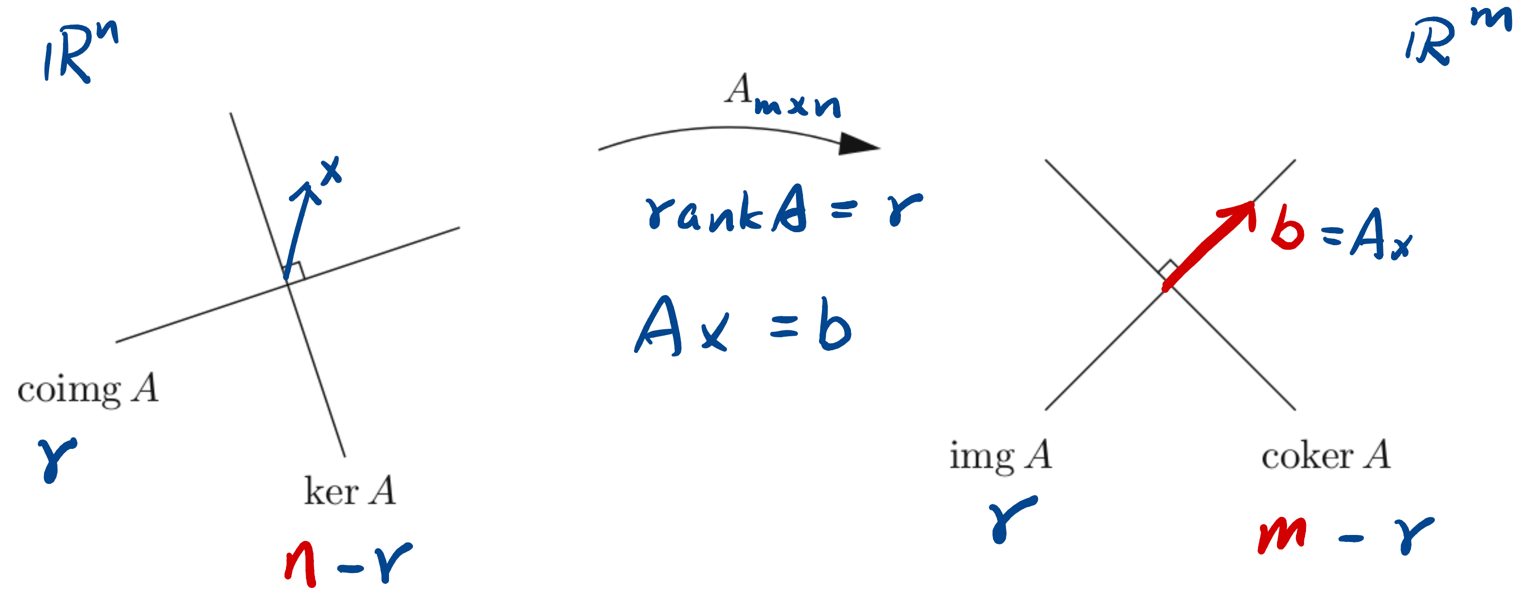
$$(\ker A)^{\perp} = \left((\operatorname{coimg} A)^{\perp}\right)^{\perp}$$
$$= \operatorname{coimg} A.$$

Similarly, applying the same reasoning to A^T, we find that

Fact 7: Let A be any real $m \times n$ matrix. Then

$$\operatorname{img} A = (\operatorname{coker} A)^{\perp} \quad (\text{and } \operatorname{coker} A = (\operatorname{img} A)^{\perp}).$$

Fact 8: [Fredholm alternative]
The linear system $A\mathbf{x} = \mathbf{b}$ has a solution (that is, it is compatible) $\Leftrightarrow b \perp \operatorname{coker} A$



Example 5. Find the compatibility condition on the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$. ($b \perp \operatorname{coker} A$)

1. Find $\operatorname{coker} A \,(= \ker A^T)$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[③-①]{②-2①} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{③-②} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solving $A^T\vec{x} = 0$, Then $\vec{x} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix}$, $z \in \mathbb{R}$.

A basis of coker A = $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

2. Let $b = (b_1, b_2, b_3)^T$

$$0 = \left\langle b, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

Then $\boxed{-b_1 - b_2 + b_3 = 0.}$

is compatibility condition. #

Remark: The "same" compatibility condition can also be obtained by using Gaussian Elimination to solve the augment system $(A|\mathbf{b})$.]

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 3 & 2 & b_3 \end{array}\right).$$

$$\xrightarrow{③-①} \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_3-b_1 \end{array}\right)$$

$$\xrightarrow{③-②} \left(\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3-b_1-b_2 \end{array}\right)$$

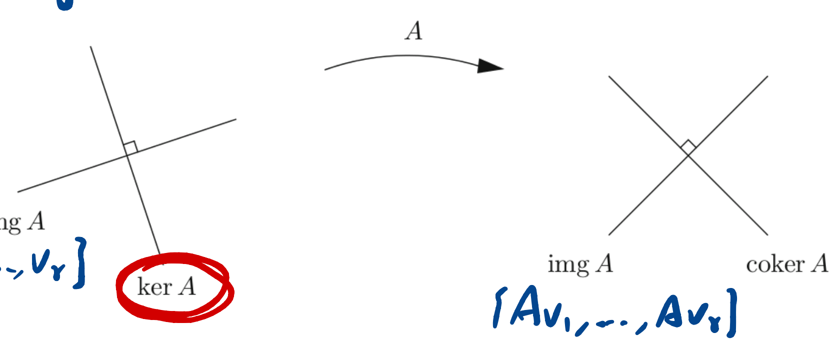
Then $Ax=b$ has solution(s) if $b_3-b_1-b_2=0$

Fact 9: If $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is a basis of coimg A, then

$$\{A\mathbf{v}_1, \cdots, A\mathbf{v}_r\} \text{ is a basis of img } A.$$

NOTE $r = \dim(\text{coimg} A) = \dim(\text{img} A)$.

It's sufficient to check $\{Av_1, \dots, Av_r\}$ are l. indep



Set $c_1Av_1 + \cdots + c_rAv_r = 0$. Claim: $c_1 = \cdots = c_r = 0$.

$$0 = A(c_1v_1 + \cdots + c_rv_r)$$

$$\Rightarrow c_1v_1 + \cdots + c_rv_r \in \ker A \cap \text{coimg} A = \{0\}.$$

$$\Rightarrow c_1v_1 + \cdots + c_rv_r = 0 \Rightarrow c_1 = \cdots = c_r = 0.$$

since $\{v_1, \dots, v_r\}$ l. indep.

We also have

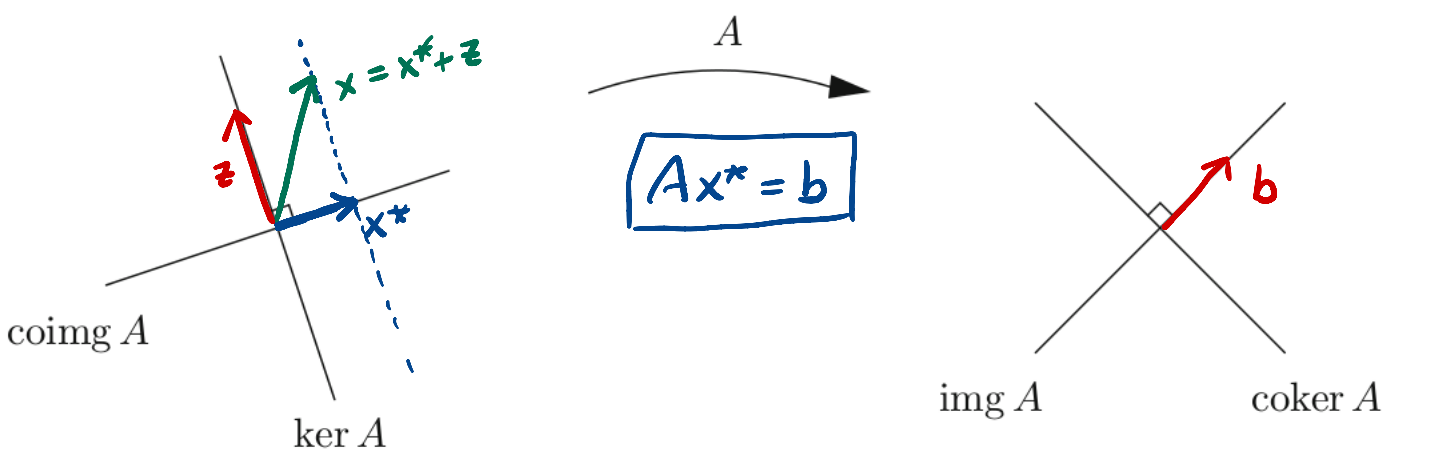
Fact 10: A compatible linear system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \in \operatorname{img} A$ has a unique solution $\mathbf{x}^*$ $\in \operatorname{coimg} A$ satisfying $A\mathbf{x}^* = \mathbf{b}$.

The general solution is

$$\boxed{\mathbf{x} = \mathbf{x}^* + \mathbf{z}}$$

where $\mathbf{x}^* \in \operatorname{coimg} A$ and $\mathbf{z} \in \ker A$. Then $\mathbf{x}^*$ has the **smallest Euclidean norm** of all the solutions to $A\mathbf{x} = \mathbf{b}$.

[To see this:]



1. $A x = A(x^* + z) = A x^* + A z$

$\qquad\qquad\qquad\qquad\qquad = b + 0$

Then $Ax = b$. (ie, x is a solution)

2. $\|x\|_2^2 = \|x^* + z\|_2^2 = \|x^*\|_2^2 + 2\cancel{\langle x^*, z\rangle} + \|z\|_2^2$ $\quad 0$

$\qquad\qquad\qquad\qquad\qquad \geq \|x^*\|_2^2$

Then $\|x\|_2 \geq \|x^*\|$, which means x^* has smallest Euclidean norm.

To find the solution of minimum Euclidean norm, that is, $\mathbf{x}^*$:

(1) Using **Gaussian Elimination** to find the general solution $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$.

(2) Finding the basis $v_1, \cdots, v_\ell$ for $\ker A$, and then using the conditions $v_j^T \mathbf{x} = 0$.

Example 6. Find the solution of minimum Euclidean norm $\mathbf{x}^*$ of the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$ and $\mathbf{b} = (1, 0, 0)^T$. $(x^* \perp \ker A)$.

1. Find general solutions.

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \xrightarrow{③ - 2②} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = -z$

$x = 1 + 2z$.

General solutions are $\left\{ \begin{pmatrix} 1+2z \\ -z \\ z \end{pmatrix} \middle| z \in \mathbb{R} \right\}$.

2. Find a basis for $\ker A$: $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$.

3. $0 = \left\langle \begin{pmatrix} 1+2z \\ -z \\ z \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

$0 = 2(1 + 2z) + z + z = 6z + 2.$

$\underline{z = -1/3}$. plug it in general sol.

to get $x^* = \begin{pmatrix} 1 + 2(-\frac{1}{3}) \\ 1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \end{pmatrix}$. #

Chapter 7 Linearity

7.1 Linear Functions

Definition: [Linear operators]
If $L : V \to W$ is a mapping between vector spaces V and W, we say that L is **linear** if for all vectors $\mathbf{x}$ and $\mathbf{y}$ in V, and scalars c such that

$$L[c\mathbf{x}] = cL[\mathbf{x}]$$
$$L[\mathbf{x} + \mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}].$$

We call such a mapping L a **linear operator**. We call V the **domain** for L, and W the **codomain**.

We may also say L is a *linear function*, or a *linear map* (or mapping), or a *linear transformation*. They all refer to the same properties.

Properties:

- For any scalars c and d and any vectors $\mathbf{x}$ and $\mathbf{y}$ in V,

$$L[c\mathbf{x} + d\mathbf{y}] = cL[\mathbf{x}] + dL[\mathbf{y}]$$

- For any scalars $c_1, \cdots, c_n$ and any vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ in V, then

$$L[c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n] = c_1L[\mathbf{x}_1] + \cdots + c_nL[\mathbf{x}_n].$$

- $L[\mathbf{0}] = \mathbf{0}$ (the $\mathbf{0}$ on the left is the zero element in V; the $\mathbf{0}$ on the right is the zero element in W).

Poll Question 1: Let A be a $m \times n$ matrix. Then coker A is orthogonal to img A.

A) Yes
B) No